Convex Optimization

Fundamentals and Applications in Statistical Signal Processing

João Mota

EURASIP/UDRC Summer School 2019

Heriot-Watt University

Optimization Problems

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- $x \in \mathbb{R}^n$: optimization variable
- $f : \mathbb{R}^n \to \mathbb{R}$: cost function (or objective)
- $\Omega \subset \mathbb{R}^n$: constraint set

Example: Polynomial Fitting

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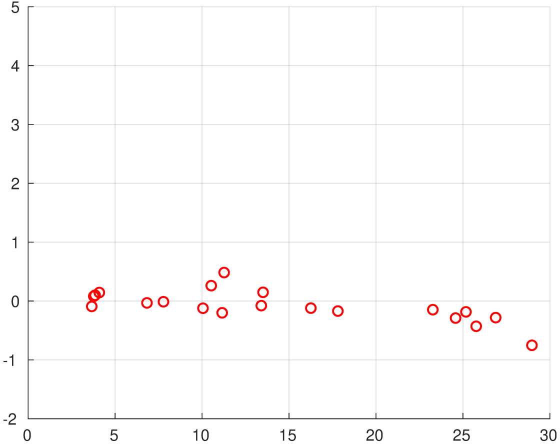

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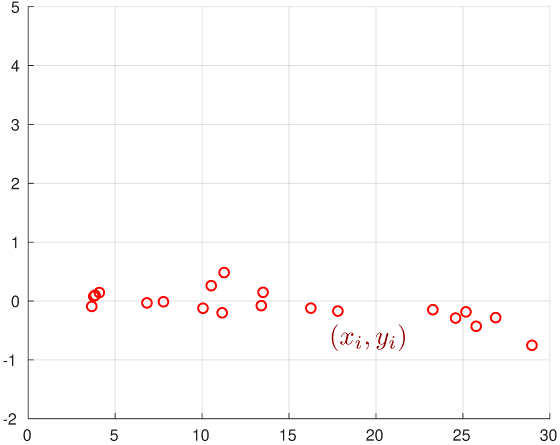

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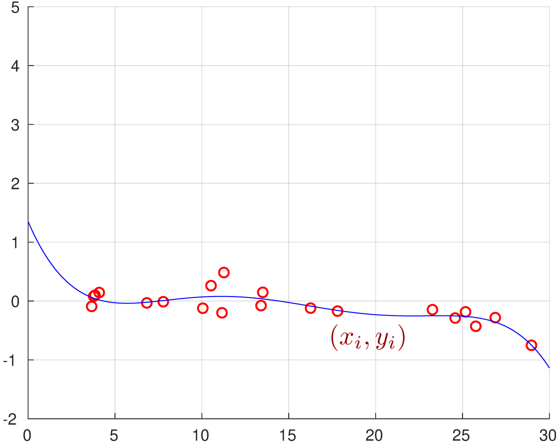

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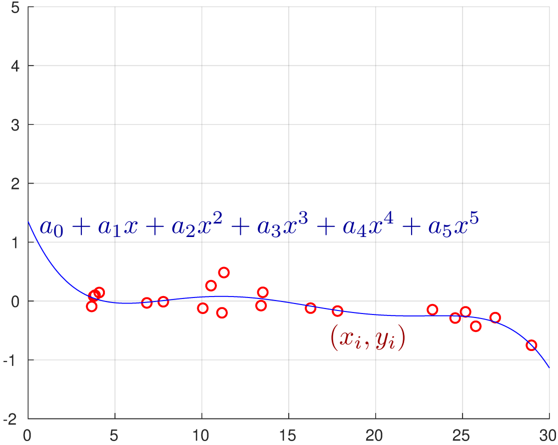

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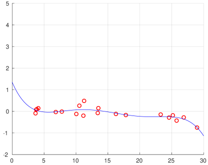

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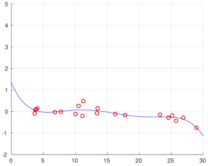

Polynomial of order $k = 5$:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

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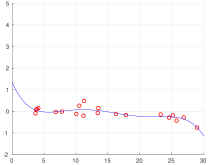

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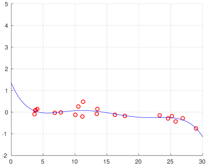

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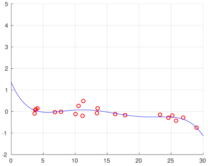

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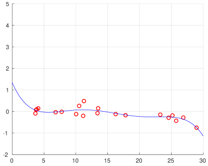

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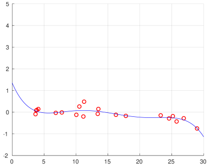

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$$\underset{a \in \mathbb{R}^5}{\text{minimize}} \quad f(a) = \left\| y - Xa \right\|_2^2 \qquad \nabla f(a^\star) = 0 \iff X^\top X a^\star = X^\top y$$

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What if there are outliers?

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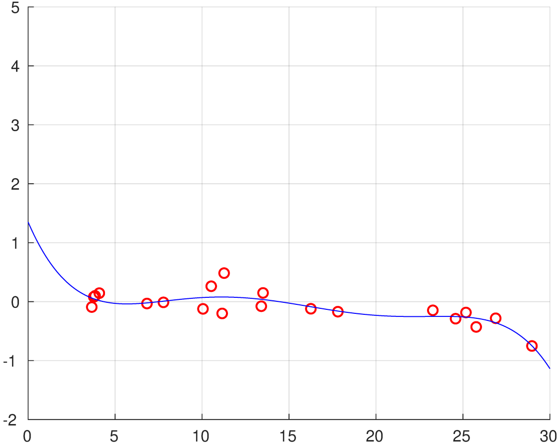

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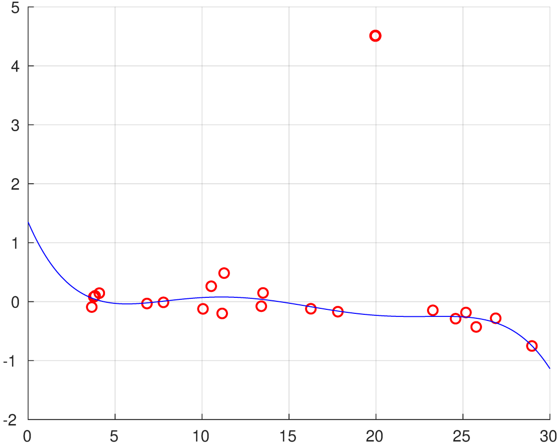

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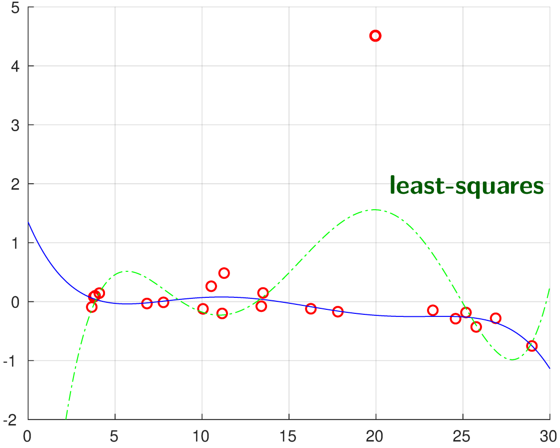

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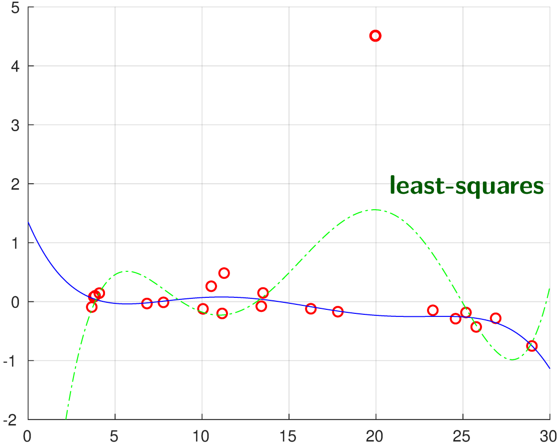

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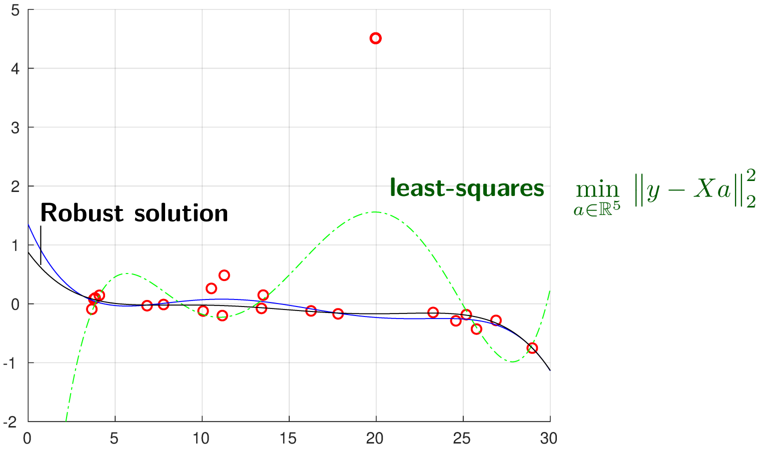

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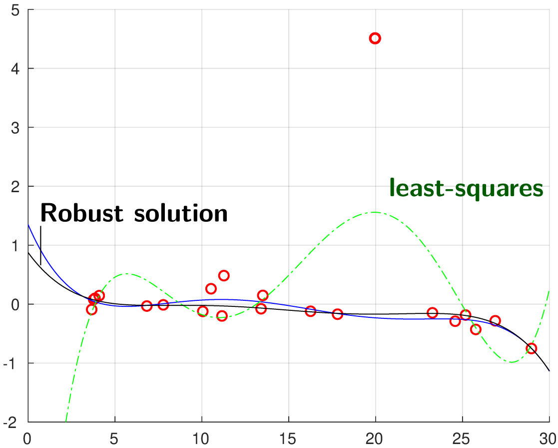

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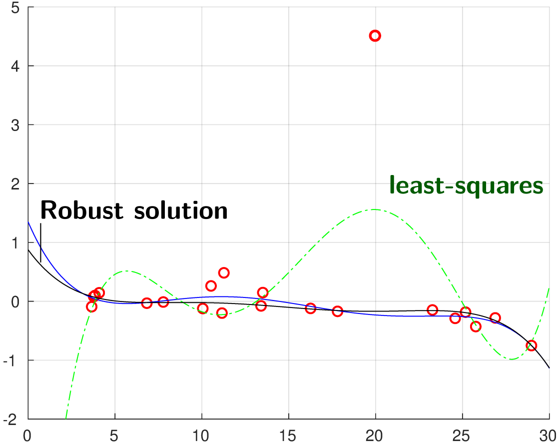

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no closed-form

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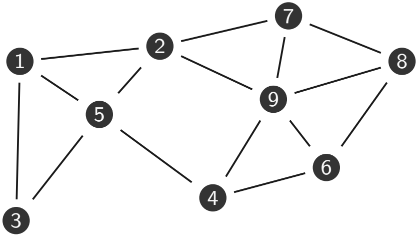

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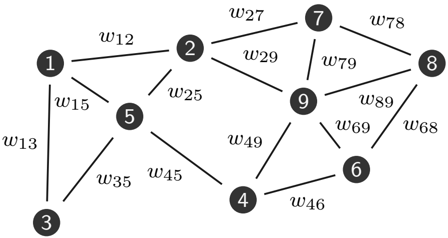

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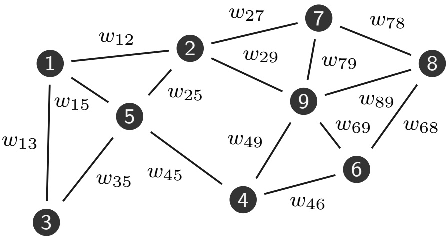

Cut: set of edges whose removal splits the graph into two

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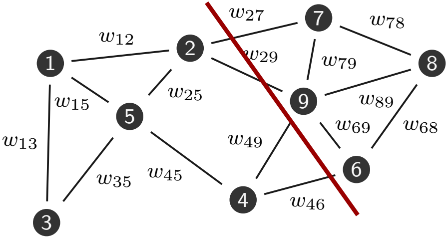

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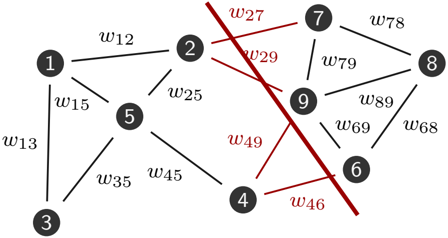

value of cut: $w_{27} + w_{29} + w_{49} + w_{46}$

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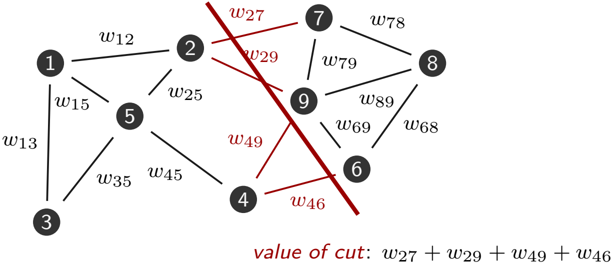

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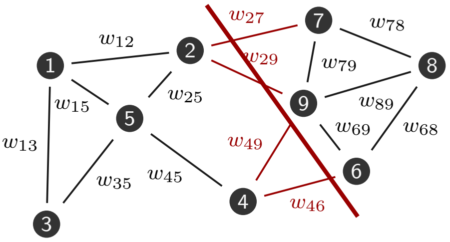

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Difficulty of optimization problems

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- Combinatorial, NP-Hard, requires exhaustive search ***(hard)***

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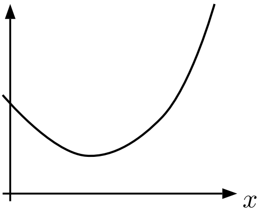
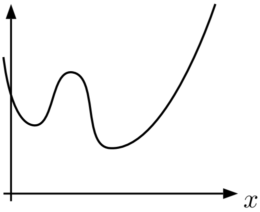


nonconvex



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- Many algorithms for *nonconvex optimization* use convex surrogates

Convex problems

Hierarchical classification (specialized solvers):

Convex problems

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LP

linear programming

Convex problems

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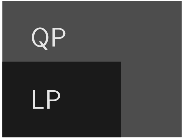

quadratic programming

linear programming

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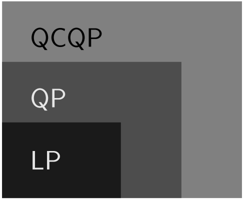

quadratically constrained QP

quadratic programming

linear programming

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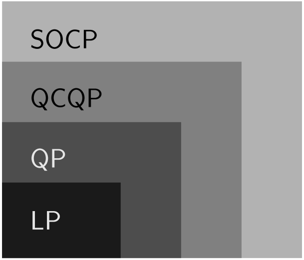

second-order cone programming

quadratically constrained QP

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linear programming

Convex problems

Hierarchical classification (specialized solvers):

semidefinite programming

second-order cone programming

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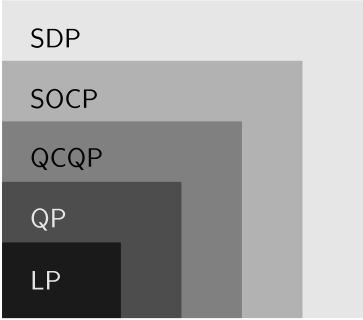

semidefinite programming

second-order cone programming

quadratically constrained QP

quadratic programming

linear programming

Other classifications:

differentiable vs. *nondifferentiable* programming

unconstrained vs. *constrained* programming

Outline

Convex sets

Identifying convex sets

Examples: geometrical sets and filter design constraints

Convex functions

Identifying convex functions

Relation to convex sets

Optimization problems

Convex problems, properties, and problem manipulation

Examples and solvers

Statistical estimation

Maximum likelihood & maximum a posteriori

Nonparametric estimation

Hypothesis testing & optimal detection

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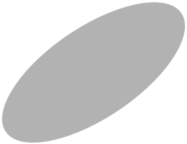


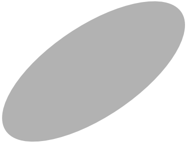
convex

nonconvex

Convex sets

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \Omega \end{array}$$

convex set



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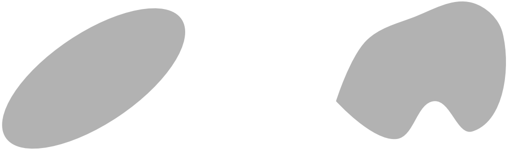

convex ***nonconvex***

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$$(1-\alpha)x + \alpha y \in C\,, \qquad \text{for all } 0 \le \alpha \le 1.$$

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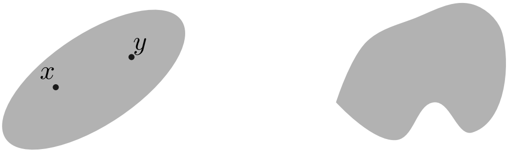

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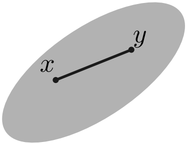

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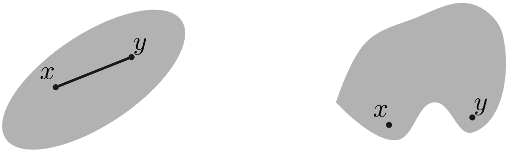

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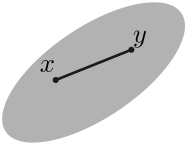

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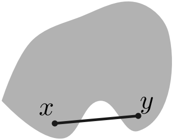

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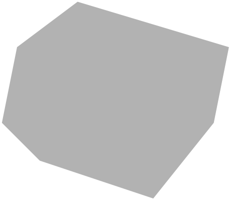

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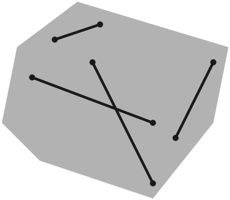

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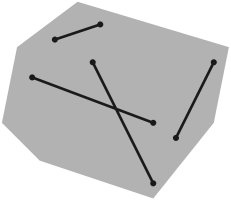

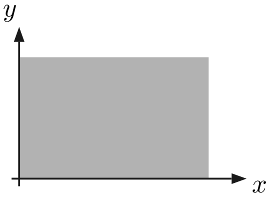

Examples of nonconvex sets

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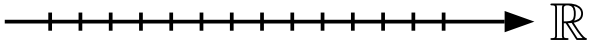

discrete sets

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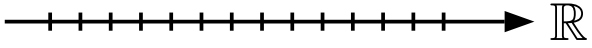

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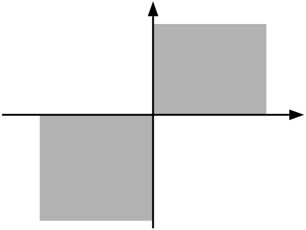

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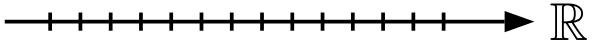

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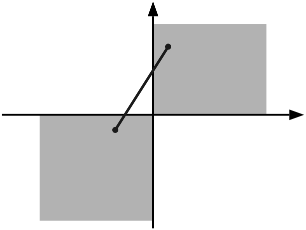

How to identify convex sets?

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vocabulary $+$ **grammar**

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simple sets

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simple sets — *operations preserving convexity*

Simple sets

Simple sets

Hyperplanes

$$\mathcal{H}_{a,b} = \left\{ x \in \mathbb{R}^n \,:\, a^\top x = b \right\}$$

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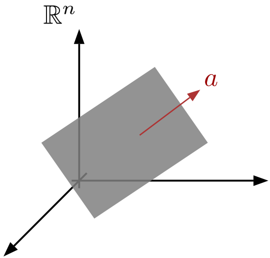

Simple sets

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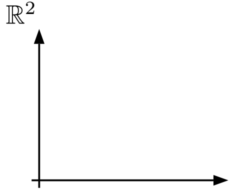

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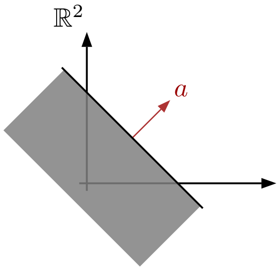

Simple sets

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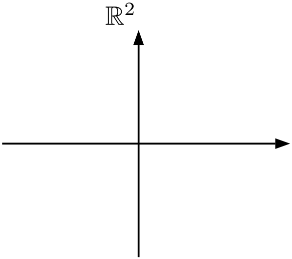

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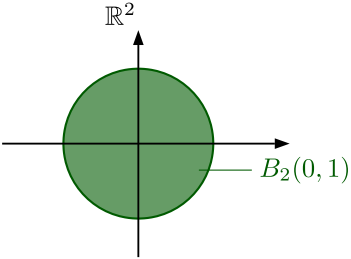

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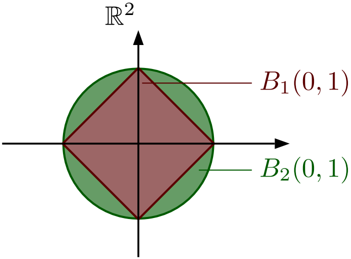

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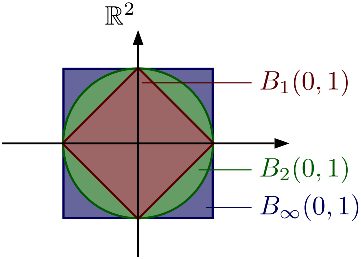

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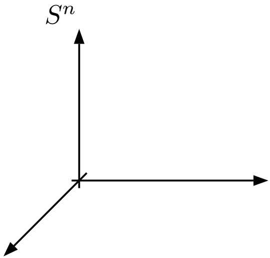

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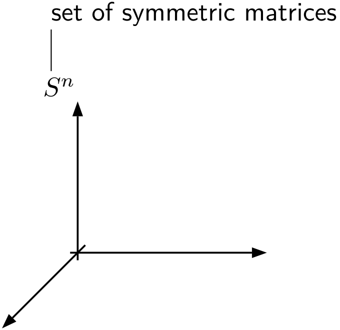

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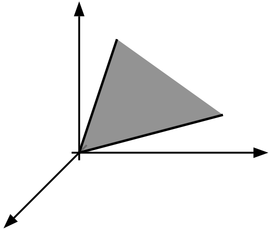
set of symmetric matrices

S^n



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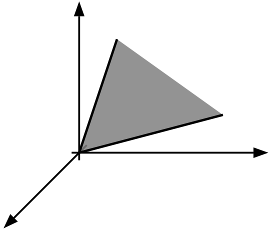

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$$X \succeq 0_{n\times n} \quad \iff \quad \lambda_{\min}(X) \geq 0 \quad \iff \quad v^\top X v \geq 0\,, \quad \forall v$$

How to identify convex sets?

vocabulary + **grammar**

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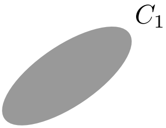

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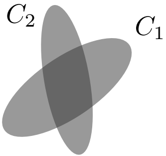

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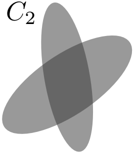

Intersection

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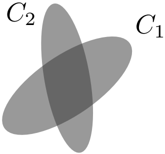

Intersection

$C_1, C_2, \ldots, C_m$: convex

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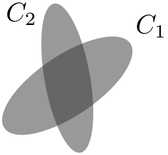

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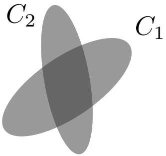

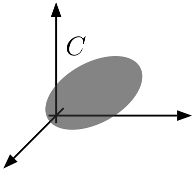

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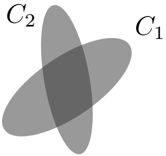

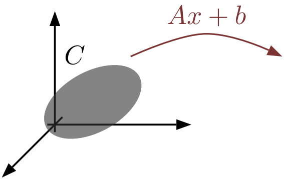

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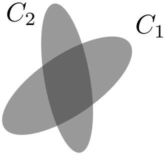

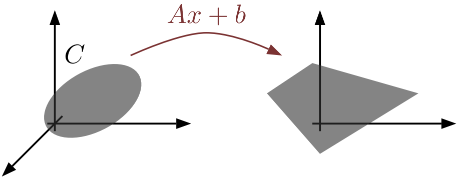

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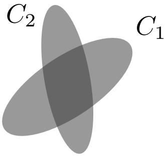

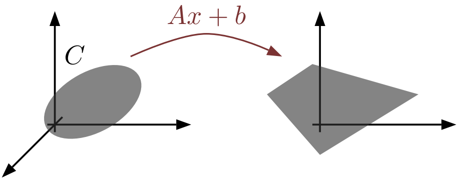

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Affine operations

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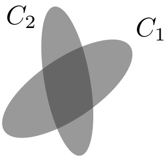

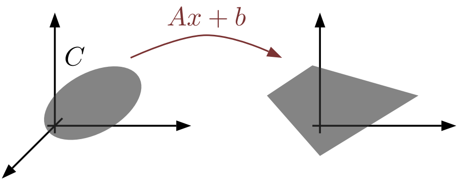

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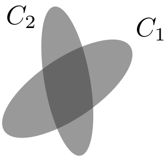

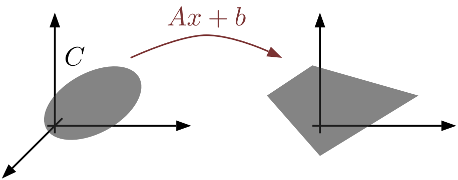

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Example

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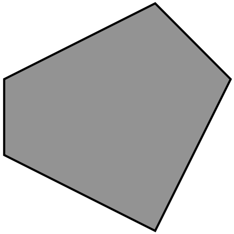

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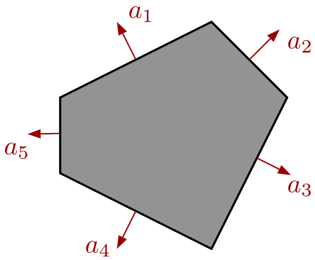

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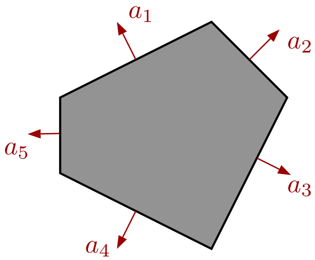

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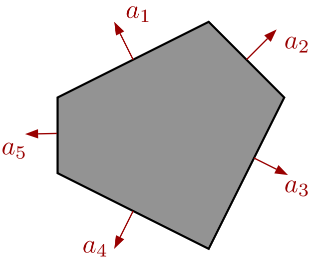

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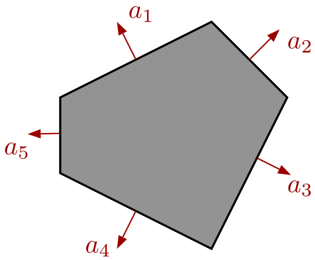

Example

Ellipsoids

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Ellipsoids $\left(A \succ 0\right)$

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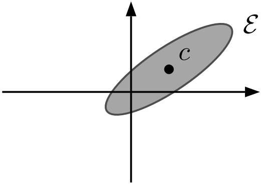

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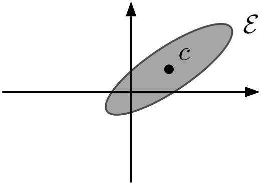

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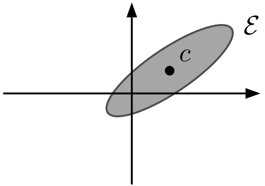

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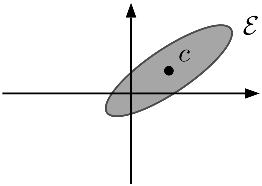

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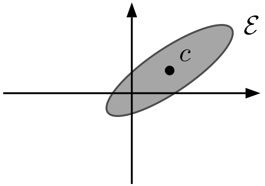

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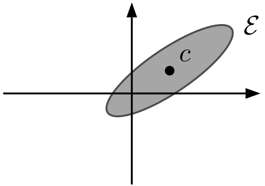

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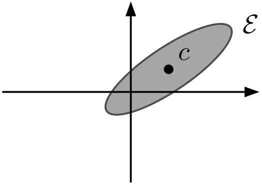

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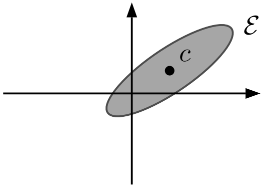

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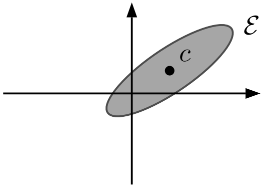

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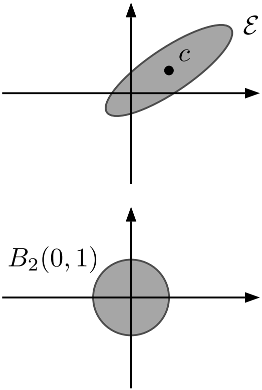

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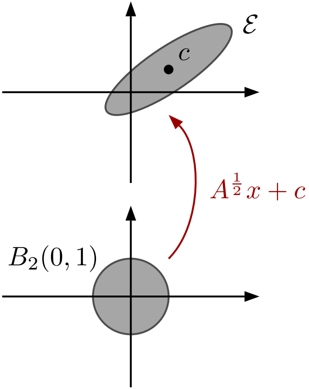

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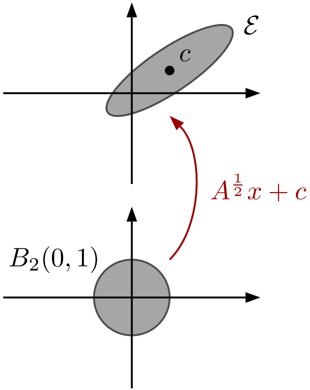

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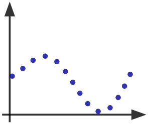

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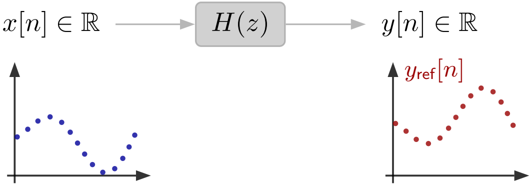

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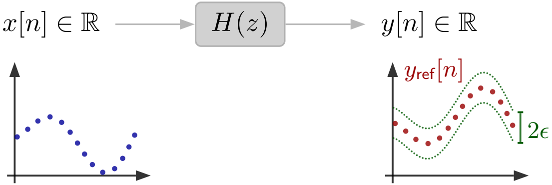

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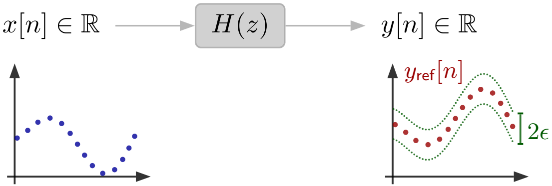

Goal: design $H(z)$ such that $\max_n \left|y[n] - y_{\text{ref}}[n]\right| \leq \epsilon$ for a fixed $x[n]$

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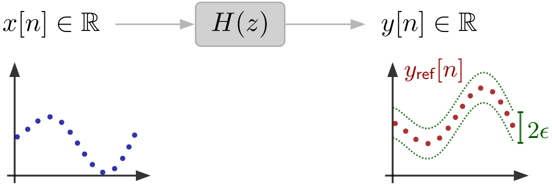

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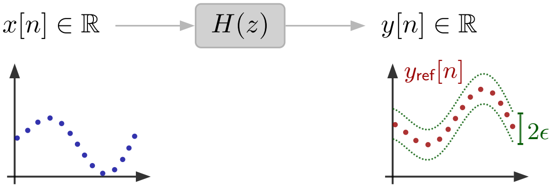

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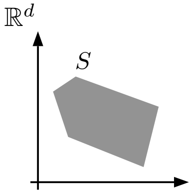

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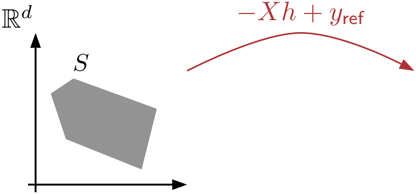

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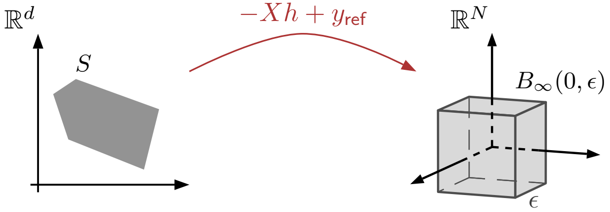

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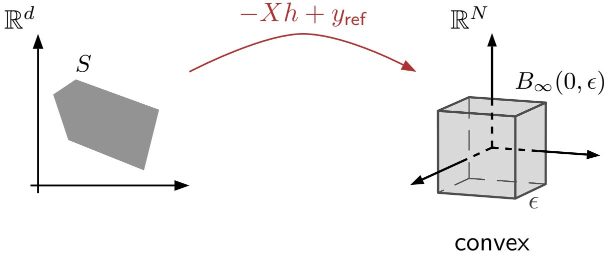

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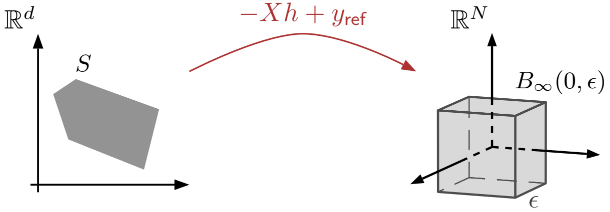

convex $\qquad \Longleftarrow \qquad$ convex

Outline

Convex sets

Identifying convex sets

Examples: geometrical sets and filter design constraints

Convex functions

Identifying convex functions

Relation to convex sets

Optimization problems

Convex problems, properties, and problem manipulation

Examples and solvers

Statistical estimation

Maximum likelihood & maximum a posteriori

Nonparametric estimation

Hypothesis testing & optimal detection

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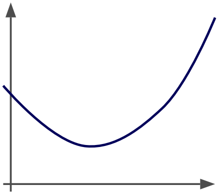

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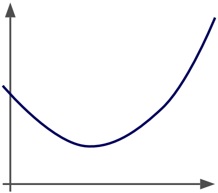

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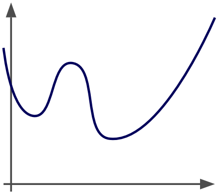

nonconvex

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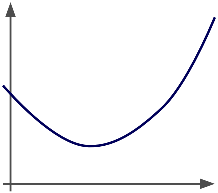

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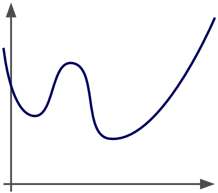

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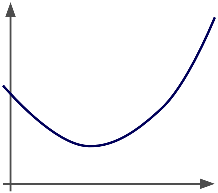

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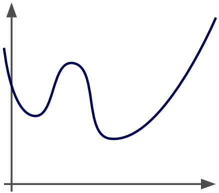

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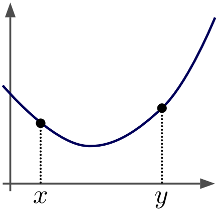

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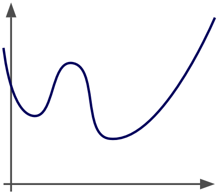

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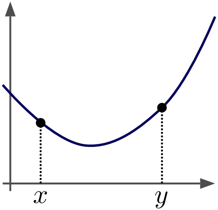

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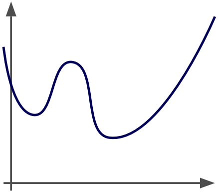

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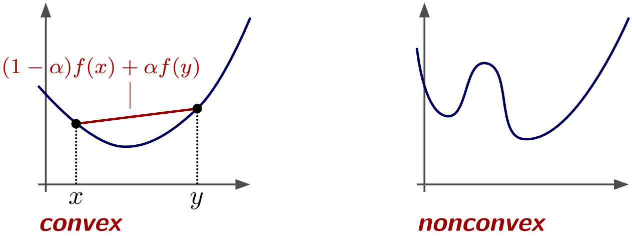

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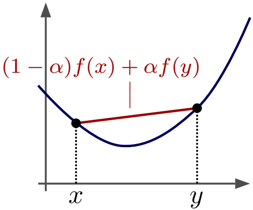

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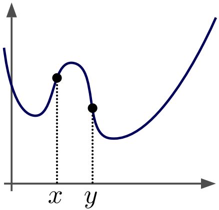

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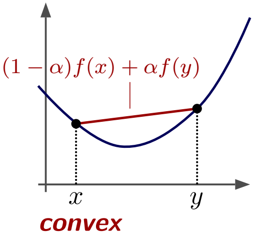

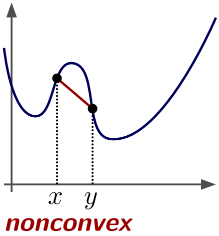

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How to identify convex functions?

vocabulary $+$ **grammar**

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vocabulary	+	**grammar**
definition		*operations preserving convexity*
differentiability conds.		
1D convexity		

Convexity under differentiability

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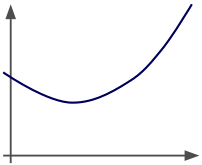

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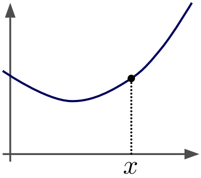

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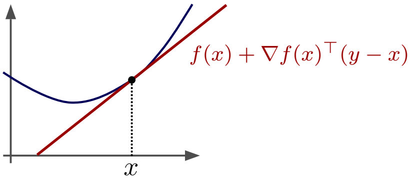

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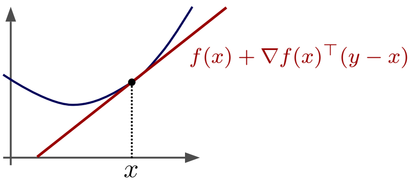

- When f is twice-differentiable,

$$\nabla^2 f(x) \succeq 0\,, \quad \forall_{x\in\mathsf{dom}\,f}$$

Examples

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Norms $f(x) = \|x\|$.

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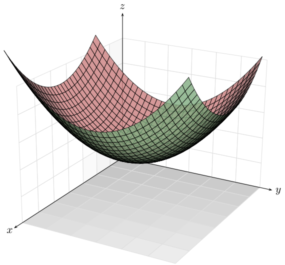

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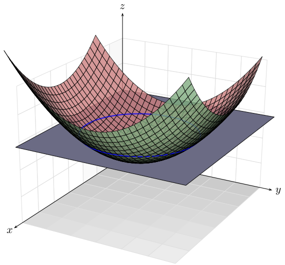

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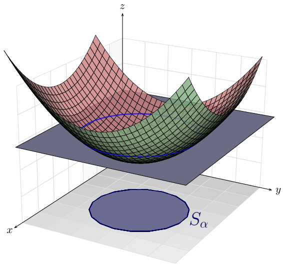

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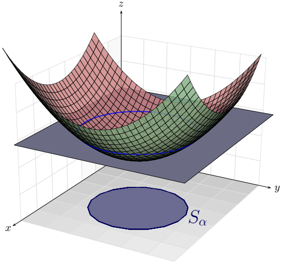

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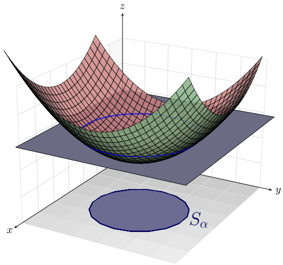

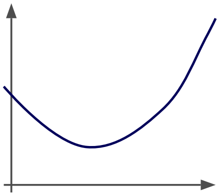

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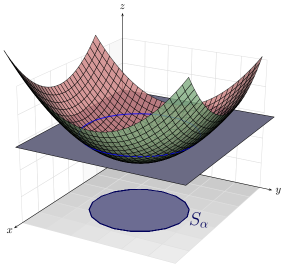

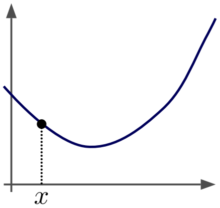

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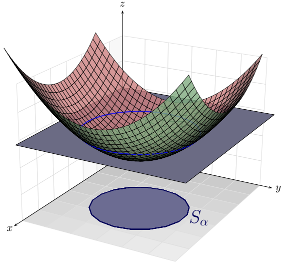

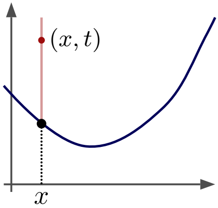

Equivalent definitions of convexity

$$f\big((1-\alpha)x+\alpha y\big) \le (1-\alpha)f(x)+\alpha f(y)\,, \qquad \forall_{x,y\in\mathsf{dom}\,f}\,,\ \ \alpha\in[0,1]$$

Equivalent statements

- $g(t)=f(x+ty)\quad (g:\mathbb{R}\to\mathbb{R})$ is convex for all $x+ty\in$ dom f
- Sublevel sets $S_\alpha := \{x \,:\, f(x)\le\alpha\}$ are convex for all $\alpha\in\mathbb{R}$
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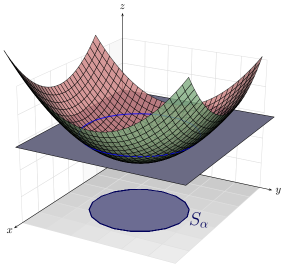

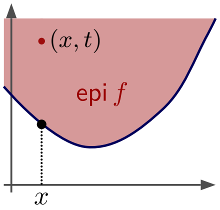

How to identify convex functions?

vocabulary	+	**grammar**
definition		*operations preserving convexity*
differentiability conds.		
1D convexity		

How to identify convex functions?

vocabulary + **grammar**

definition

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operations preserving convexity

Operations preserving convexity

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Nonnegative weighted sums

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$$\begin{cases} f_1, \ldots, f_m \ : \ \text{convex} \\ w_1, \ldots, w_m \geq 0 \end{cases}$$

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$$\begin{cases} f_1, \ldots, f_m \ : \ \text{convex} \\ w_1, \ldots, w_m \geq 0 \end{cases} \quad \Longrightarrow \quad w_1 f_1 + \cdots + w_m f_m \ : \ \text{convex}$$

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Precomposition with affine maps

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Outline

Convex sets

Identifying convex sets

Examples: geometrical sets and filter design constraints

Convex functions

Identifying convex functions

Relation to convex sets

Optimization problems

Convex problems, properties, and problem manipulation

Examples and solvers

Statistical estimation

Maximum likelihood & maximum a posteriori

Nonparametric estimation

Hypothesis testing & optimal detection

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A minimizer: $$x^\star \in \underset{x}{\text{argmin}} \; f(x) \quad \text{s.t.} \quad x \in \Omega$$

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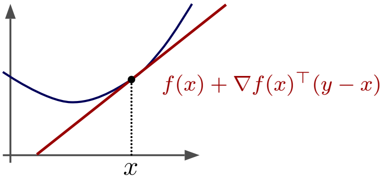

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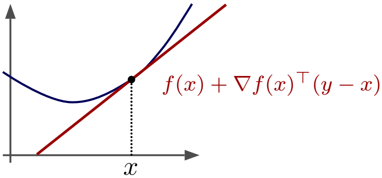

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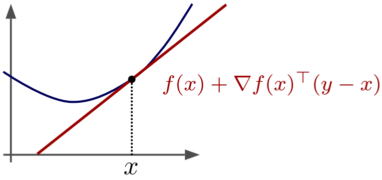

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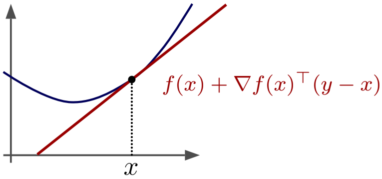

$\square$

Equivalence between optimization problems

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$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in X \end{array} \tag{P1}$$

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(P1) and (P2) are *equivalent* when

- Given a solution $x^\star$ of (P1) we can obtain a solution $y^\star$ of (P2)
- Given a solution $y^\star$ of (P2) we can obtain a solution $x^\star$ of (P1)

Examples

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$$\underset{x}{\text{maximize}} \quad f(x)$$

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if im $f \subseteq$ dom g

and $g \circ f$: convex

Air Traffic Control

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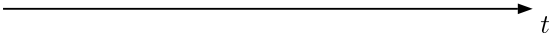

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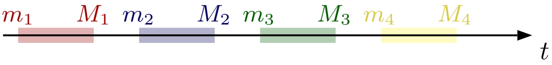

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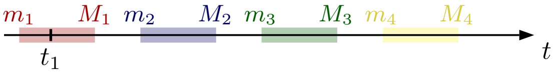

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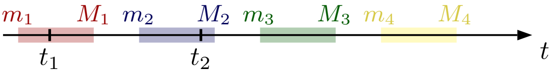

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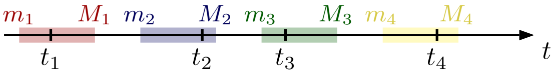

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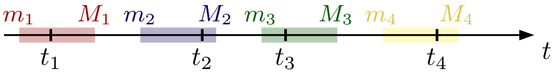

Goal: compute $t_1, \ldots, t_n$ that maximize $\min_i \; t_{i+1} - t_i$

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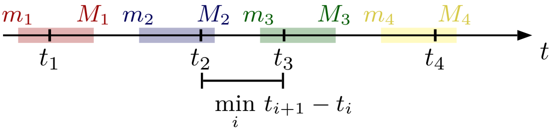

Goal: compute t_1, ..., t_n that maximize $\min_i \; t_{i+1} - t_i$

Air Traffic Control

$$
\begin{array}{ll}
\underset{t_1,\ldots,t_n}{\text{maximize}} & \min\left\{t_2 - t_1\,,\; t_3 - t_2\,,\; \ldots\,,\; t_n - t_{n-1}\right\} \\
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\end{array}
$$

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$$\iff \quad \begin{array}{ll} \underset{t_1,\ldots,t_n}{\text{minimize}} & -\min\left\{t_2 - t_1\,,\ t_3 - t_2\,,\ \ldots\,,\ t_n - t_{n-1}\right\} \\ \text{subject to} & m_i \le t_i \le M_i\,, \quad i = 1, \ldots, n \end{array}$$

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Convex

Air Traffic Control

$$\begin{array}{ll} \underset{t_1,\ldots,t_n}{\text{minimize}} & \max\left\{t_1 - t_2\,,\ t_2 - t_3\,,\ \ldots\,,\ t_{n-1} - t_n\right\} \\ \text{subject to} & m_i \le t_i \le M_i\,, \quad i = 1,\ldots,n \end{array}$$

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$$\iff \quad \begin{array}{ll} \underset{t_1,\ldots,t_n,s}{\text{minimize}} & s \\ \text{subject to} & t_1 - t_2 \le s \\ & \quad \vdots \\ & t_{n-1} - t_n \le s \\ & m_i \le t_i \le M_i\,, \quad i = 1,\ldots,n \end{array}$$

Air Traffic Control

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$$A = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 1 & -1 & \cdots & 0 & 0 & -1 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 1 & -1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -m_1 \\ M_1 \\ \vdots \\ -m_n \\ M_n \end{bmatrix}$$

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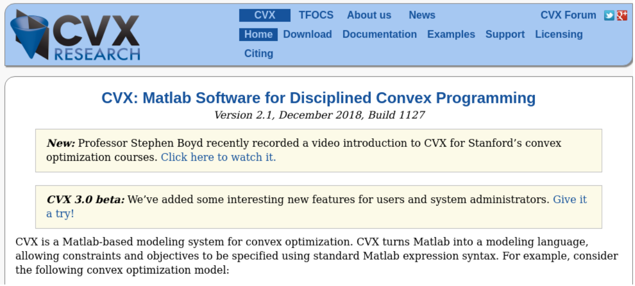
CVX (cvxr.com/cvx) manipulates and solves *convex* problems



Air Traffic Control

```
cvx_begin
    variables t1 t2 t3 t4 t5;
    maximize( min([t2-t1, t3-t2, t4-t3, t5-t4]) );
    subject to
        1 <= t1 <= 2;
        3 <= t2 <= 4;
        5 <= t3 <= 6;
        7 <= t4 <= 8;
        9 <= t5 <= 10;
cvx_end
```

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```
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```

$$(t_1^\star,\ t_2^\star,\ t_3^\star,\ t_4^\star,\ t_5^\star) = (1,\ 3.25,\ 5.5,\ 7.75,\ 10)$$

Portfolio Optimization

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Portfolio Optimization

Different investment strategies

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$$
\begin{array}{ll}
\underset{x}{\text{minimize}} & \mathsf{Var}\big(s(x)\big) \\
\text{subject to} & x \geq 0_n \\
& 1_n^\top x = T \\
& \mathbb{E}\big[s(x)\big] \geq s_{\min}
\end{array}
$$

Portfolio Optimization

Different investment strategies

- Minimize variance while guaranteeing minimum expected return $s_{\min}$:

$$\begin{array}{llcll} \underset{x}{\text{minimize}} & \mathsf{Var}\big(s(x)\big) & \Longleftrightarrow & \underset{x}{\text{minimize}} & x^\top \Sigma x \\ \text{subject to} & x \geq 0_n & & \text{subject to} & x \geq 0_n \\ & 1_n^\top x = T & & & 1_n^\top x = T \\ & \mathbb{E}\big[s(x)\big] \geq s_{\min} & & & \mu^\top x \geq s_{\min} \end{array}$$

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Convex Quadratic Program (QP)

Portfolio Optimization

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$$\begin{array}{ll} \underset{x}{\text{maximize}} & \mathbb{E}\big[s(x)\big] - \beta\,\mathsf{Var}(s(x)) \\ \text{subject to} & x \geq 0_n \\ & 1_n^\top x = T \end{array}$$

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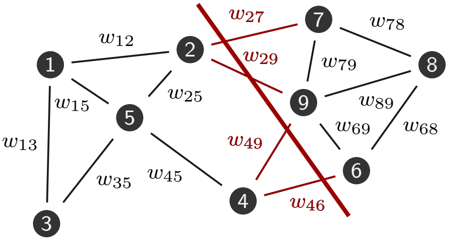
Convex QP

MAXCUT



value of cut: $w_{27} + w_{29} + w_{49} + w_{46}$

Cut: set of edges whose removal splits the graph into two

MAXCUT problem: find the cut with maximum weight

$$
\begin{array}{ll}
\underset{x \in \mathbb{R}^n}{\text{maximize}} & \displaystyle \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \frac{1 - x_i x_j}{2} \\
\text{subject to} & x_i \in \{-1, 1\}\,, \qquad i = 1, \ldots, n\,.
\end{array}
$$

$$
\begin{aligned}
p^\star = \quad \max_x \quad & \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\,\frac{1 - x_i x_j}{2} \\
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$$

$$
\begin{aligned}
= -\ \min_x \quad & \frac{1}{4}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} x_i x_j - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\right] \\
\text{s.t.} \quad & x_i^2 = 1\,, \qquad i = 1, \ldots, n
\end{aligned}
$$

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$$
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$$

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$W \in \mathbb{R}^{n\times n}$: weighted adjacency matrix

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Convex Semi-Definite Program (SDP)

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It can be shown that

$$d^\star \;\geq\; p^\star \;\geq\; \mathbb{E}[C] \;\geq\; 0.87856\, d^\star$$

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Large-scale problems & real-time solutions require tailored solvers

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For $n = 5000$ and $m = 500$,

- CVX: 56.16 s
- SPGL1: 0.82 s (tailored solver)

Optimization Algorithms

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  Block coordinate descent

  Newton (and approximations)

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And now, *neural networks!*

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- $f^\star := \min_x \ f(x) > -\infty$

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Example

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \log \sum_{i=1}^{m} \exp\left(a_i^\top x + b\right)$$

randomly generated data with $m = 500$, $n = 50$

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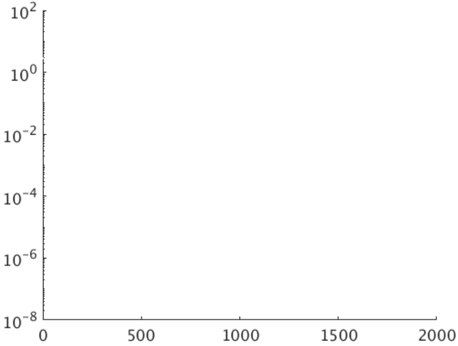
$|f(x^k) - f^\star| / |f^\star|$



iterations k

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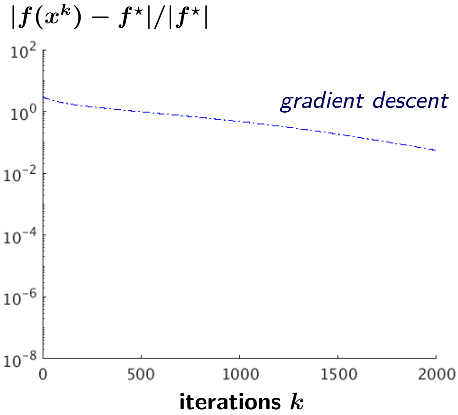

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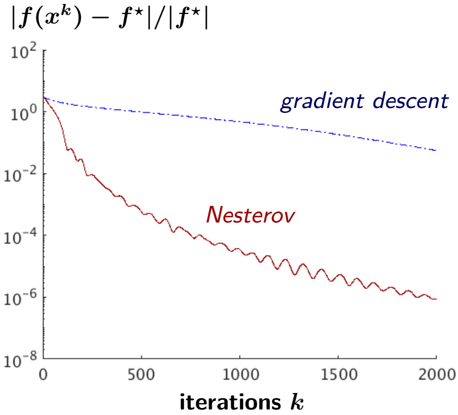

Outline

Convex sets

Identifying convex sets

Examples: geometrical sets and filter design constraints

Convex functions

Identifying convex functions

Relation to convex sets

Optimization problems

Convex problems, properties, and problem manipulation

Examples and solvers

Statistical estimation

Maximum likelihood & maximum a posteriori

Nonparametric estimation

Hypothesis testing & optimal detection

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The a_i's will denote the rows of $\quad A = \begin{bmatrix} — & a_1^\top & — \\ & \vdots & \\ — & a_m^\top & — \end{bmatrix} \in \mathbb{R}^{m \times n}$

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Convex

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Laplacian:

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feasibility problem (convex)

Example with a discrete RV

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$Y \in \mathbb{N}$: $\#$ of traffic accidents in a given period

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Goal:

Given m independent observations $\left\{\left(U^{(i)}, Y^{(i)}\right)\right\}_{i=1}^m$, estimate a and b.

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Joint probability mass function: $p_{YU}(y, u\,;\, a, b) = \frac{e^{-(a^\top u + b)}\,(a^\top u + b)^y}{y\,!}$

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Convex

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$Y \in \mathbb{R}^m$: random vector w/ density $f_Y(y)$

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$$f_X(x) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\Big(-\frac{1}{2}(x - \overline{x})^\top \Sigma^{-1}(x - \overline{x})\Big)$$

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ML and MAP allow estimating parameters of given distributions

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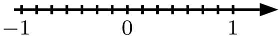

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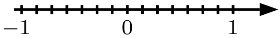

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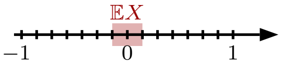

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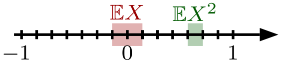

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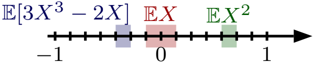

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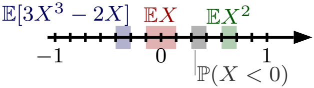

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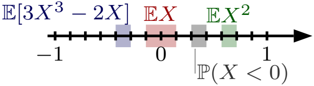

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Find a distribution satisfying these constraints & with maximum entropy

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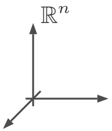

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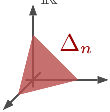

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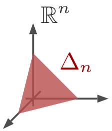

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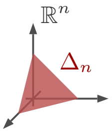

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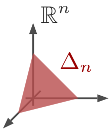

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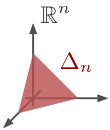

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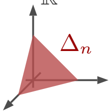

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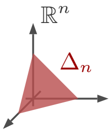

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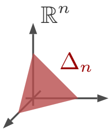

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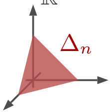

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All constraints are *linear inequalities* in p!

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Optimization problem: *(convex)*

$$\begin{aligned}
\underset{p \in \mathbb{R}^{100}}{\text{minimize}} \quad & \textstyle\sum_{i=1}^{n} p_i \log p_i \\
\text{subject to} \quad & -0.1 \le \alpha^\top p \le 0.1 \\
& 0.5 \le \beta^\top p \le 0.6 \\
& -0.3 \le \gamma^\top p \le -0.2 \\
& 0.3 \le \sigma^\top p \le 0.4
\end{aligned}$$

Nonparametric estimation

```
n = 100;
alpha = linspace(-1,1,n)';
cvx_begin
    variable p(n,1);
    minimize( -sum( entr( p ) ) );
    subject to
        p >= 0;
        ones(1, n)*p == 1;
        -0.1 <= alpha'*p <= 0.1;
        0.5 <= (alpha.^2)'*p <= 0.6;
        -0.3 <= (3*alpha.^3 - 2*alpha)'*p <= -0.2;
        0.3 <= (alpha < 0)'*p <= 0.4;
cvx_end
```

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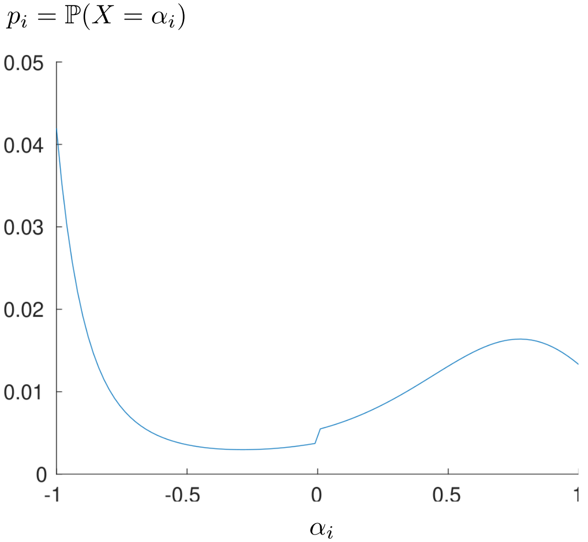

Hypothesis testing & optimal detection

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$X \in \{1, \ldots, n\}$

Hypothesis testing & optimal detection

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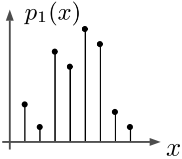

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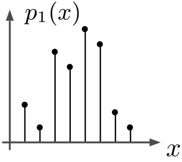

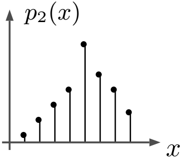

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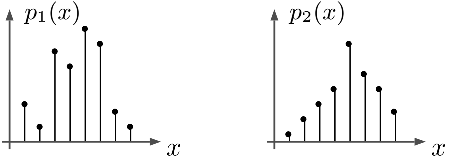

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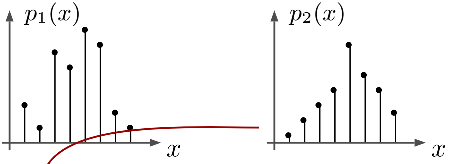

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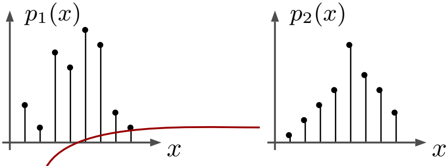

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Goal: Estimate θ based on an observation of X

Detector:

Detector: $\Psi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$

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If each t_i is a canonical vector $(0,\ldots,1,\ldots,0)$, then T is deterministic

Detection probability matrix:

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Multi-objective optimization

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Convex

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  where c_i: ith column of $C := WP^\top$

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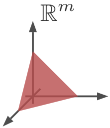

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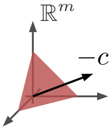

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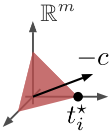

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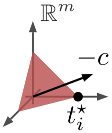

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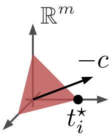

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Therefore,

$$c_i^\top t_i = t_{1i}\,W_{12}\,\mathbb{P}(X=i\,|\,\theta=2) + t_{2i}\,W_{21}\,\mathbb{P}(X=i\,|\,\theta=1)$$

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Neyman-Pearson lemma:

For each $\alpha > 0$, the likelihood-ratio test yields a (deterministic) Pareto-optimal detector.

Deterministic vs randomized detectors

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Minimax detector (random):

$$T_{\mathsf{MM}} = \begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 1/3 & 1 & 1 \end{bmatrix}$$

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Receiver operating characteristic (ROC)

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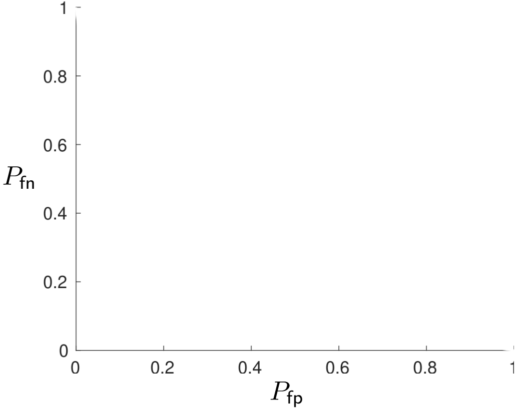

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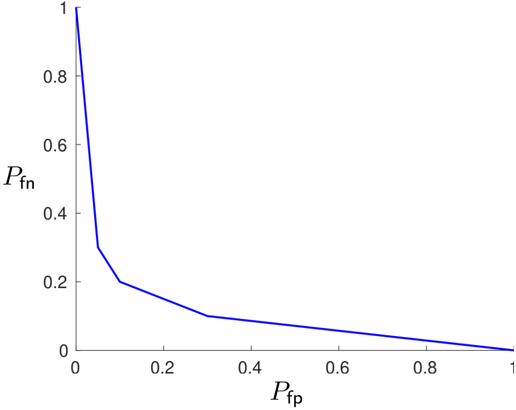

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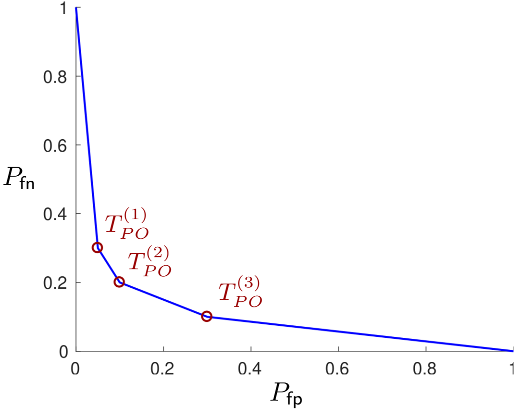

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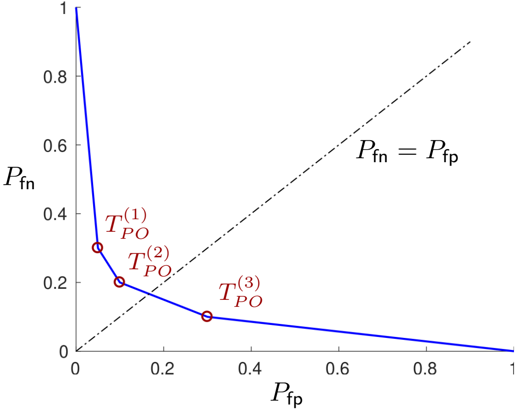

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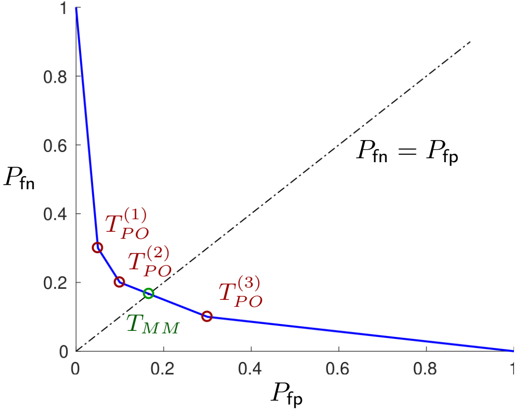

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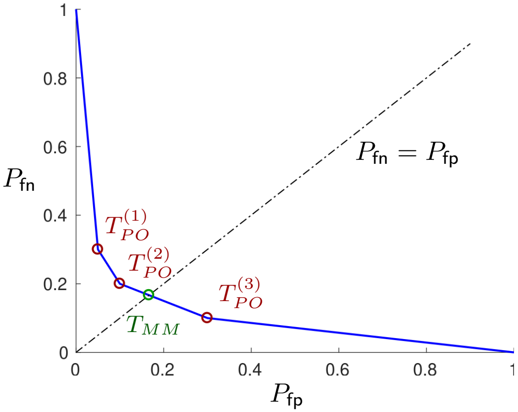

Minimax estimator has $(P_{\mathsf{fp}}, P_{\mathsf{fn}}) = (\frac{1}{6}, \frac{1}{6})$ and outperforms any deterministic estimator

Conclusions

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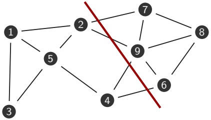

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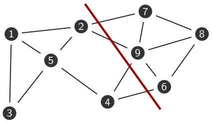

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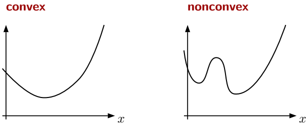

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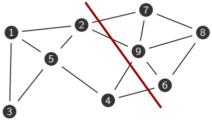

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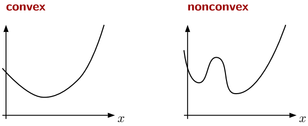

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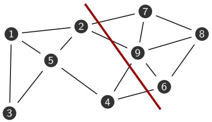

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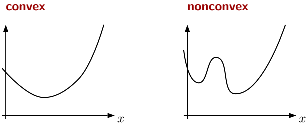

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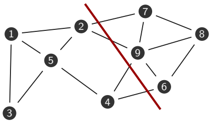

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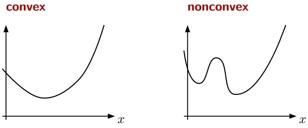

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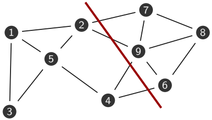

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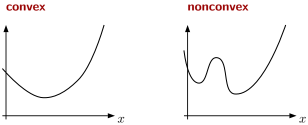
  Find parameters of distributions (ML/MAP)

  Entire distributions (nonparametric)



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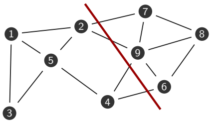

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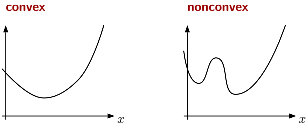
- *Didn't cover* optimality conditions & theory
- *Didn't cover* optimization algorithms



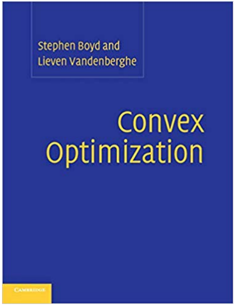
- Statistical estimation:

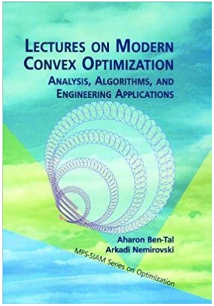
  Find parameters of distributions (ML/MAP)

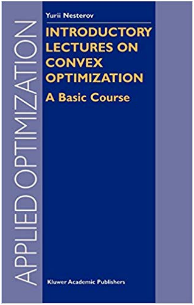
  Entire distributions (nonparametric)

- Multiple hypothesis testing via optimization

References and Resources







Lectures:

- web.stanford.edu/∼boyd/cvxbook/
- users.isr.ist.utl.pt/∼jxavier/NonlinearOptimization18799-2018
- www.seas.ucla.edu/∼vandenbe/ee236c