High-Dimensional Statistics & Sparsity

UDRC Summer School

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Heriot-Watt University, Edinburgh, UK


Motivation

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Hypothesis testing in high-dimensions

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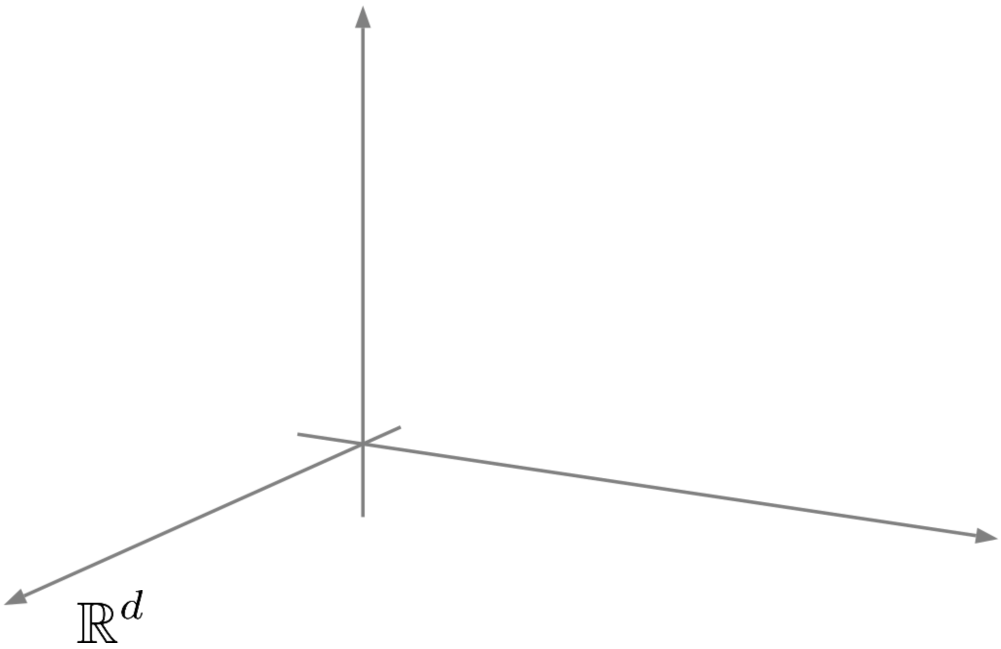

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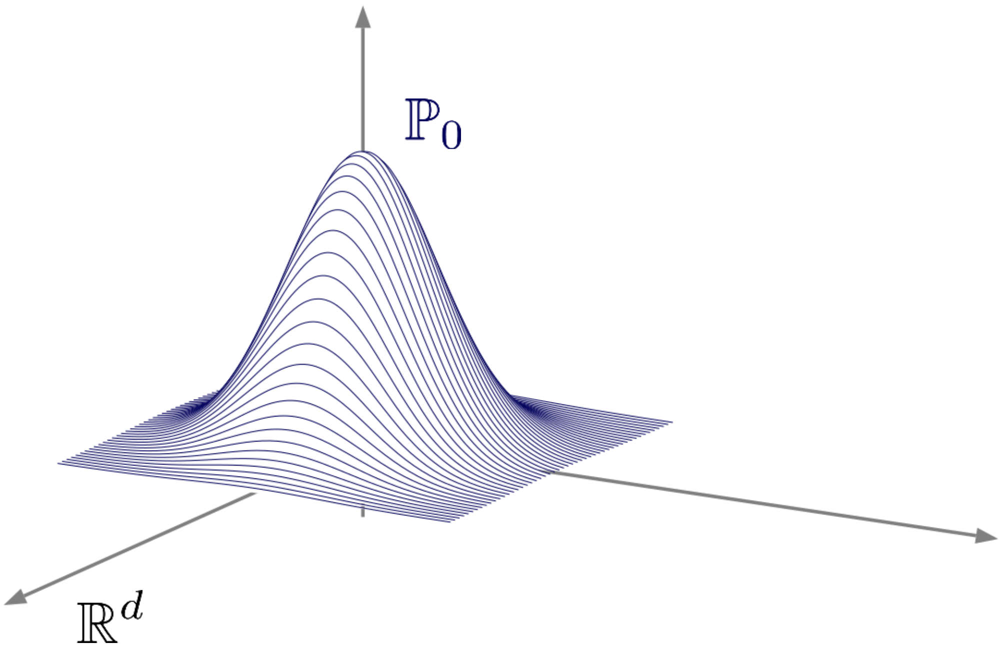

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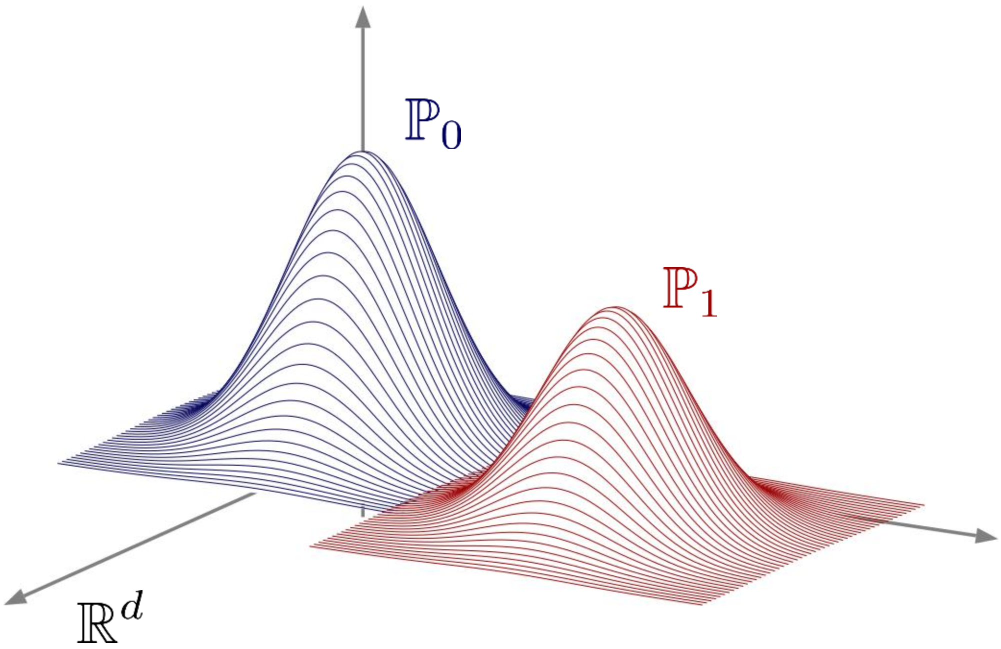

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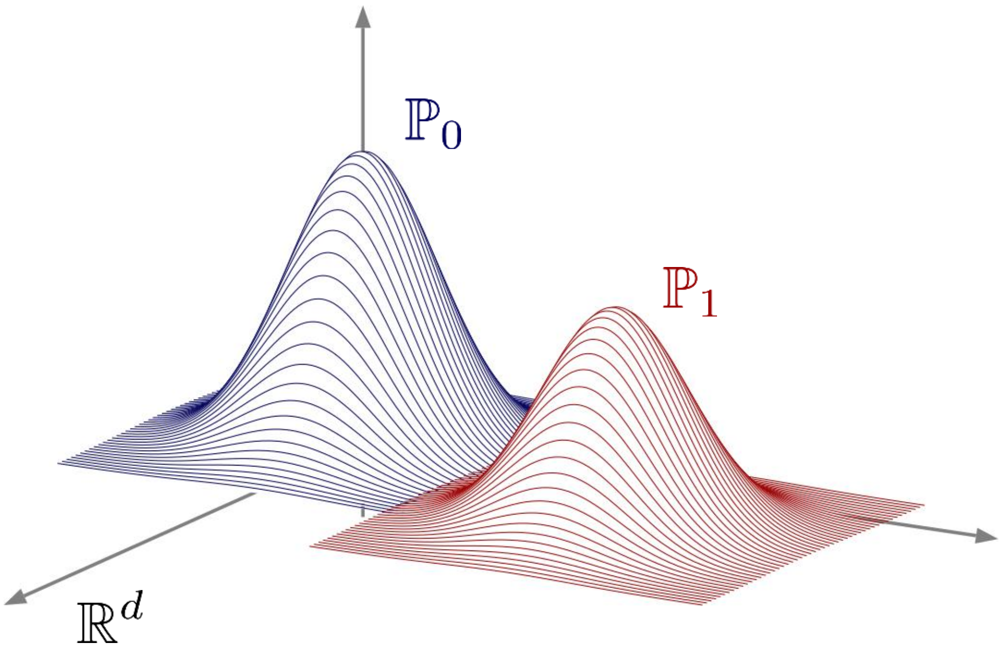

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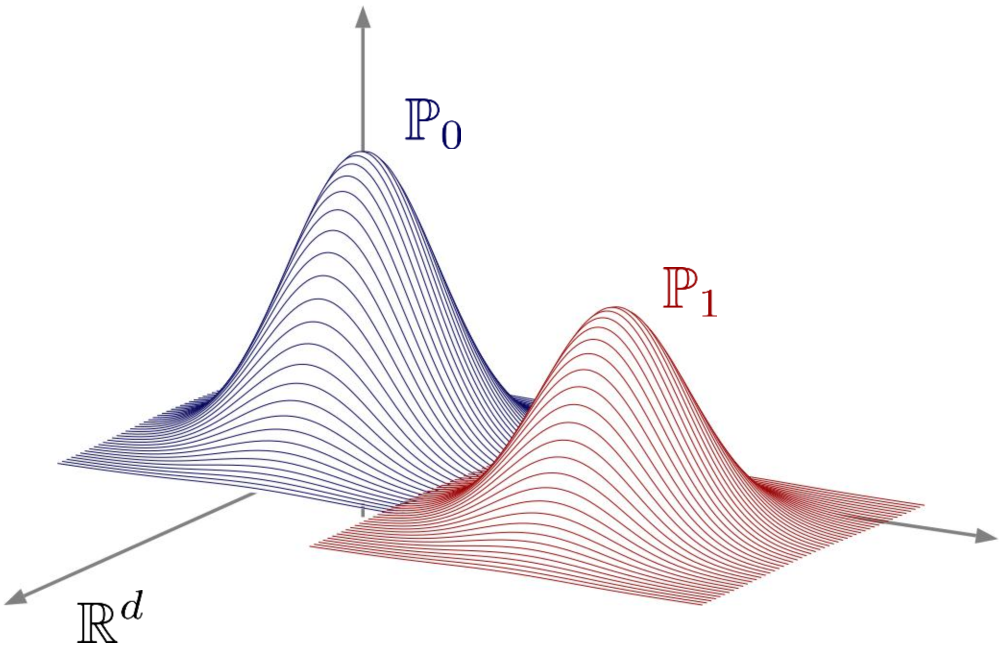

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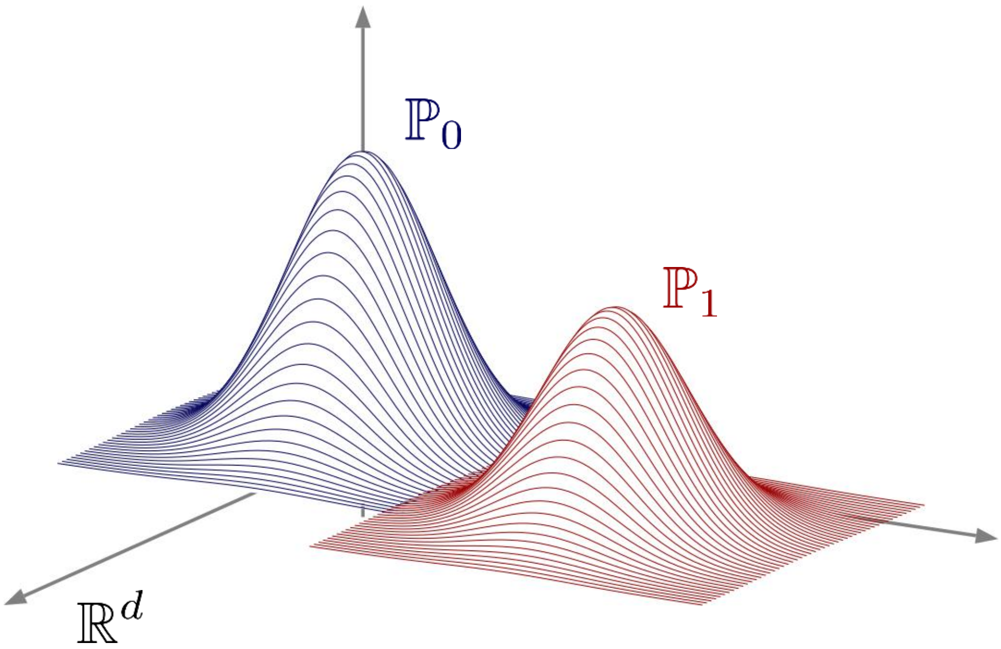

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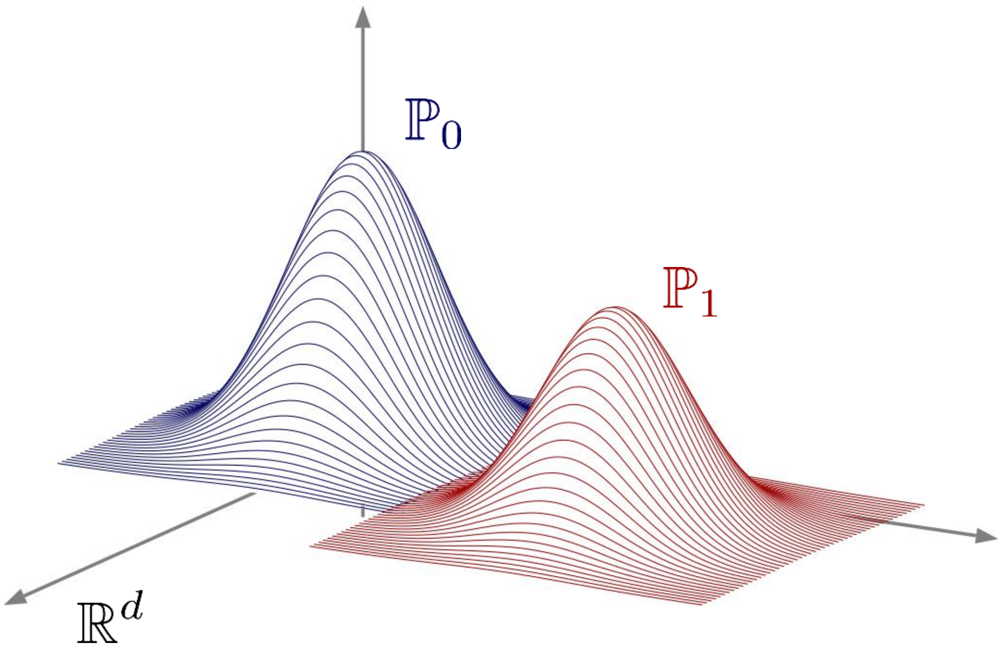

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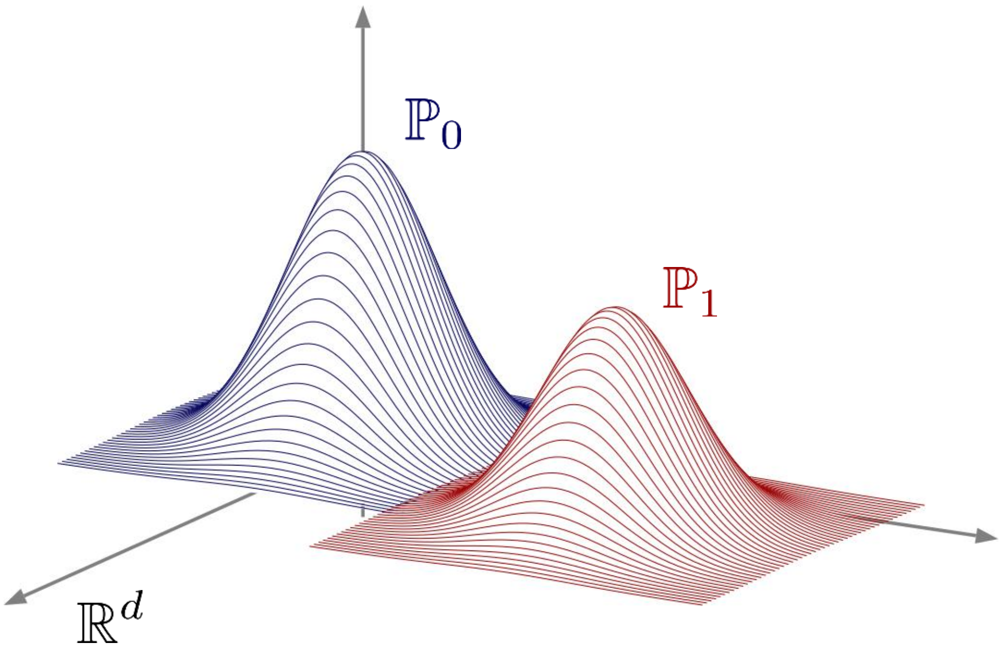

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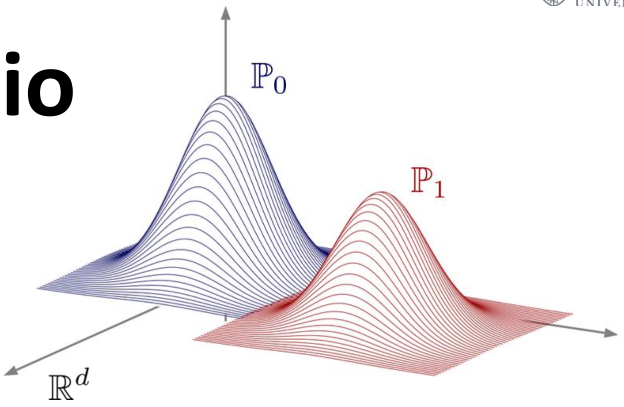
Decision Rule: Likelihood Ratio



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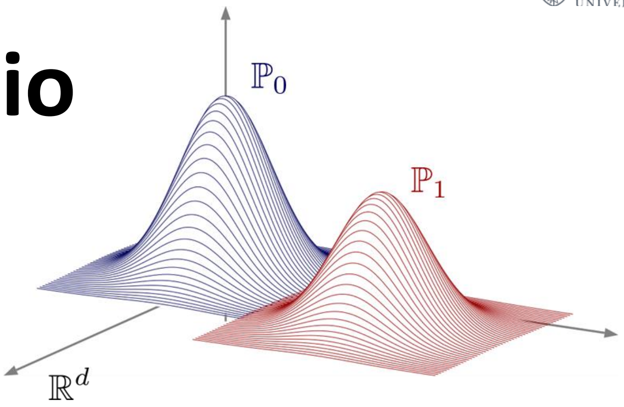

$x_1, \ldots, x_n$: i.i.d. realizations of X

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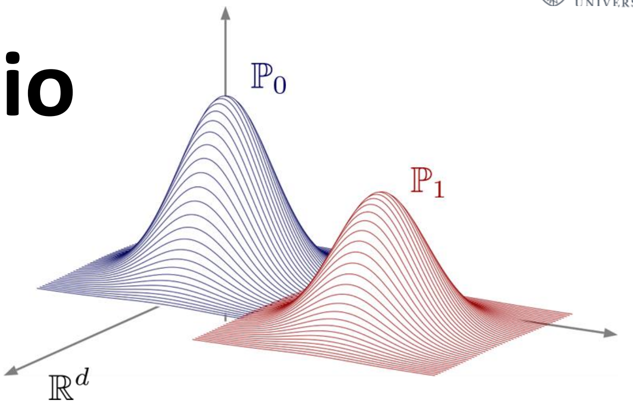

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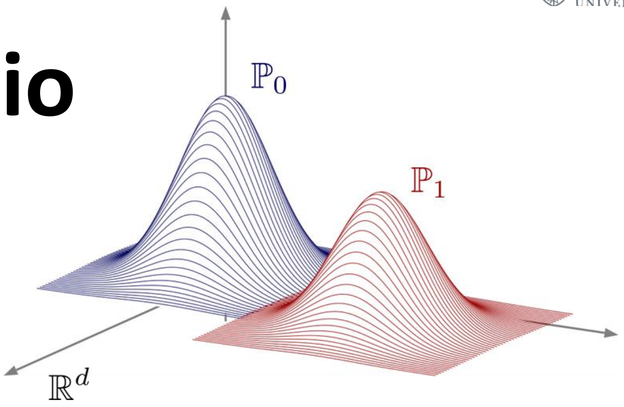

$x_1, \ldots, x_n$: i.i.d. realizations of X

For a given $T \geq 0$,

If $\dfrac{\mathbb{P}_1(x_1, \ldots, x_n)}{\mathbb{P}_0(x_1, \ldots, x_n)} > T$

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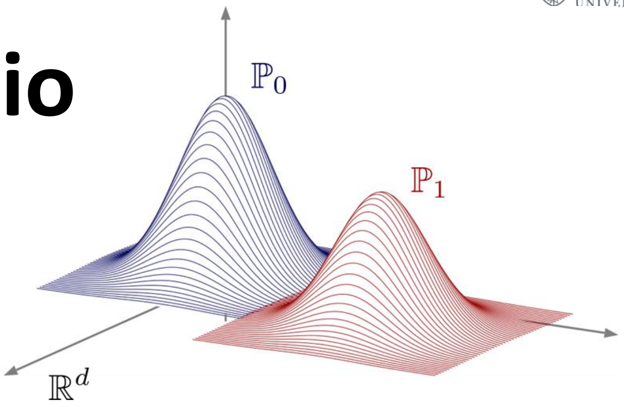

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For a given $T \geq 0$,

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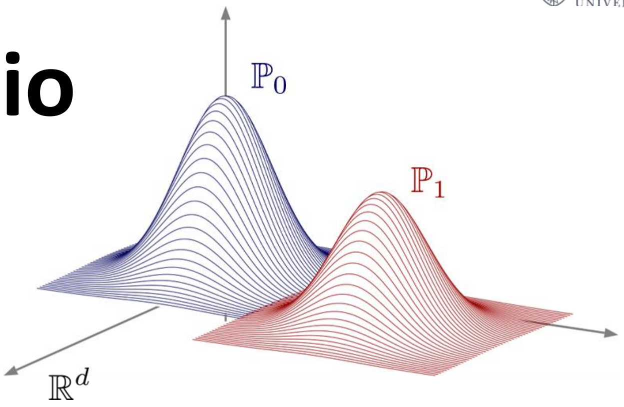

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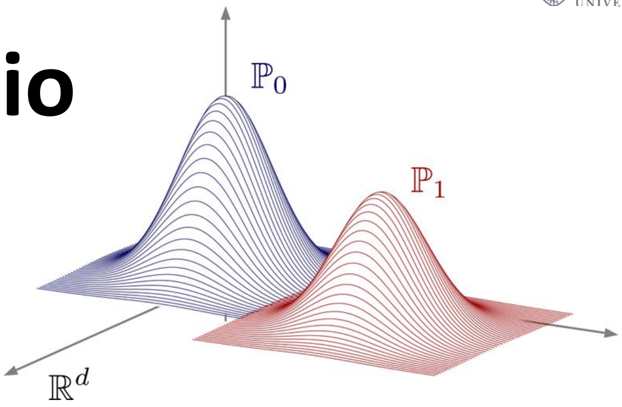

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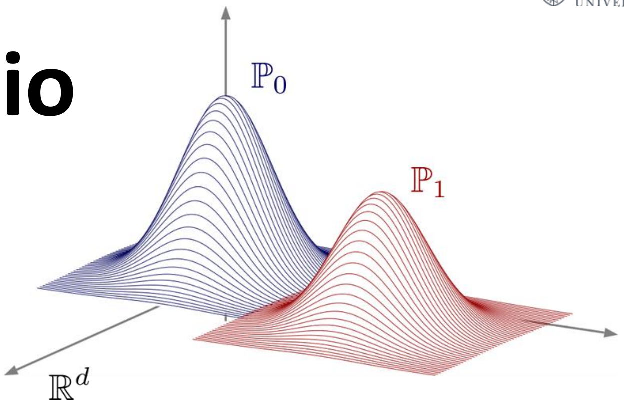

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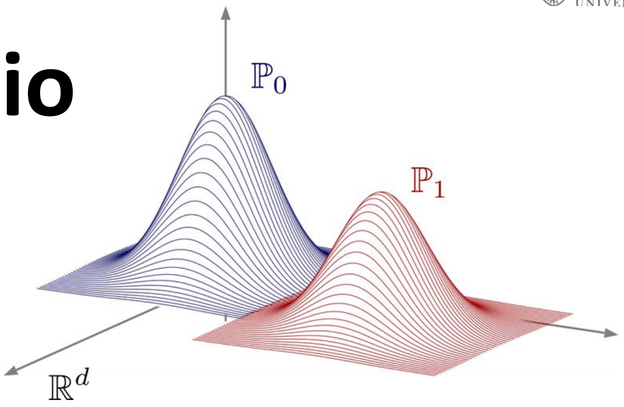

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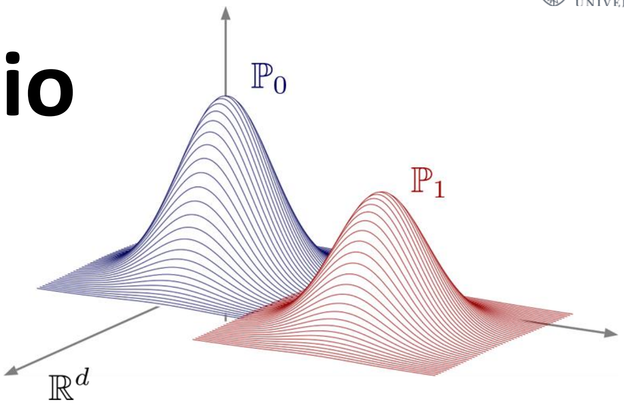

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MAP rule / minimizes risk when $T = \dfrac{\mathbb{P}(X \sim \mathbb{P}_0)}{\mathbb{P}(X \sim \mathbb{P}_1)}$

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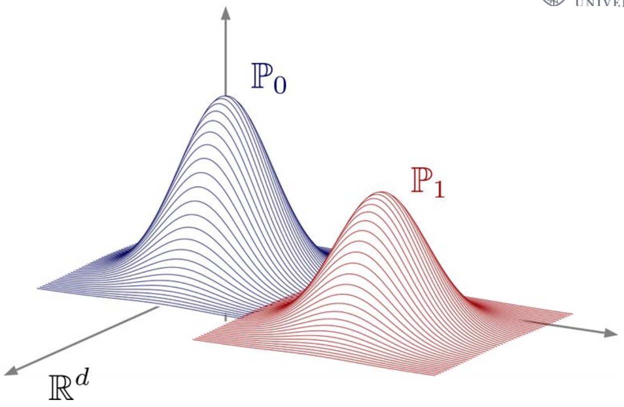
Neyman-Pearson Lemma



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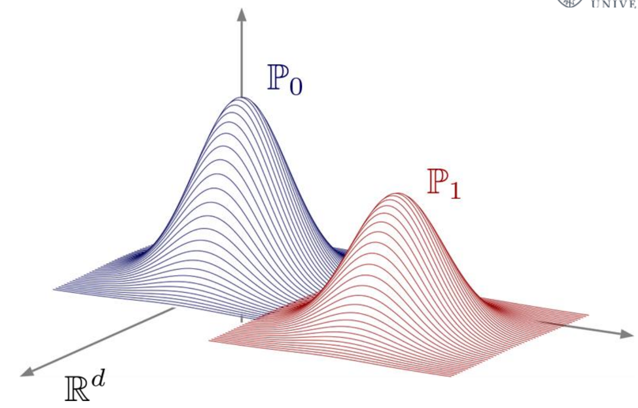

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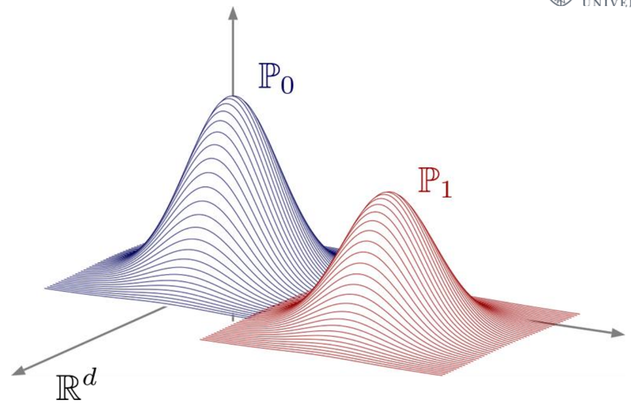

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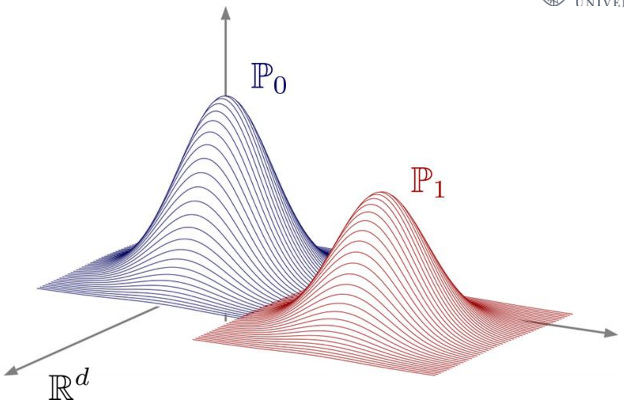

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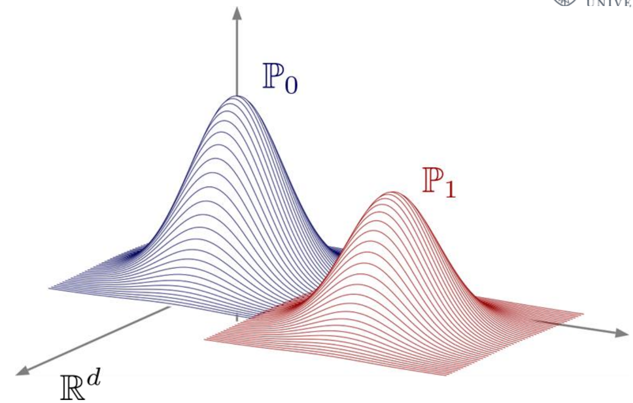

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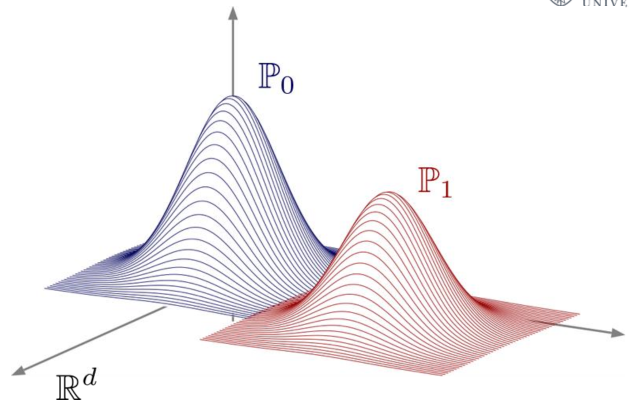

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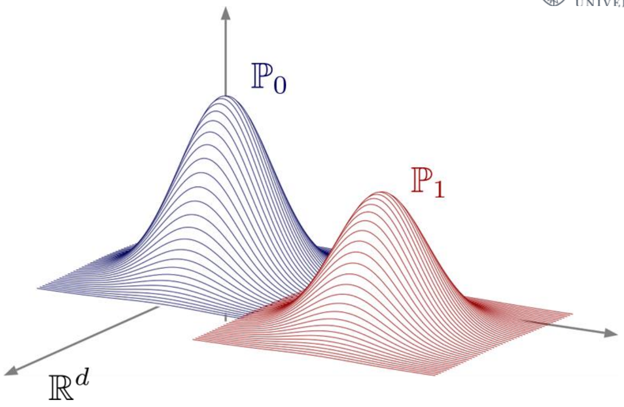

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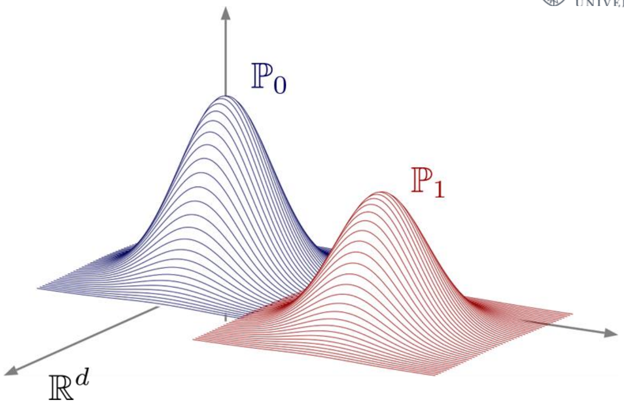

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And vice-versa.

Linear Discriminant Analysis

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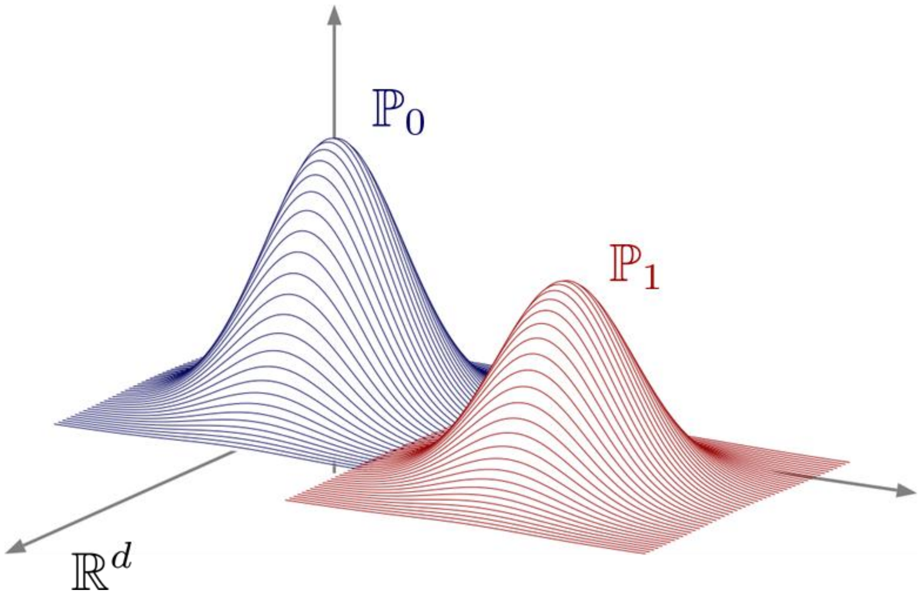

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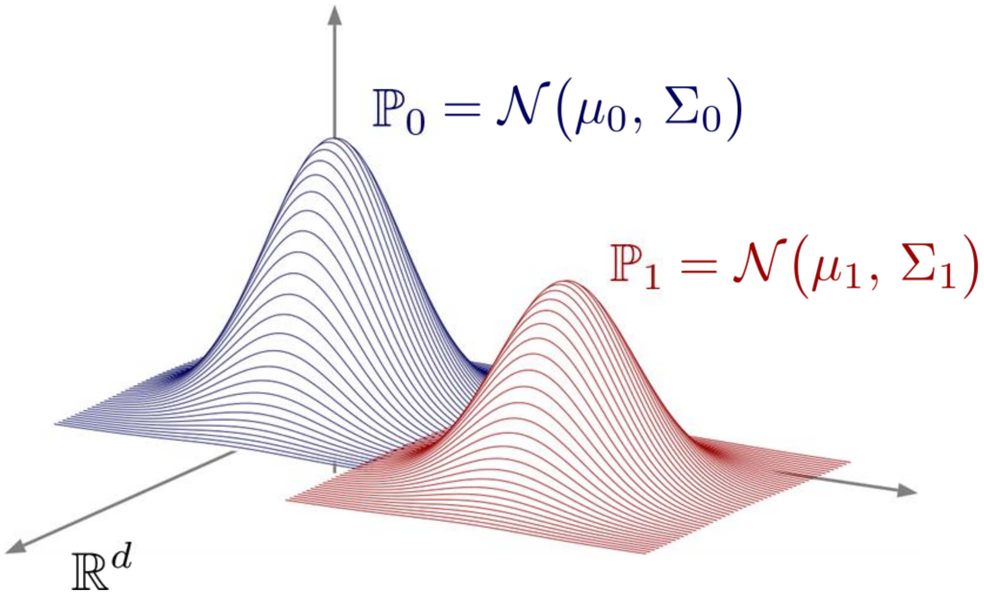

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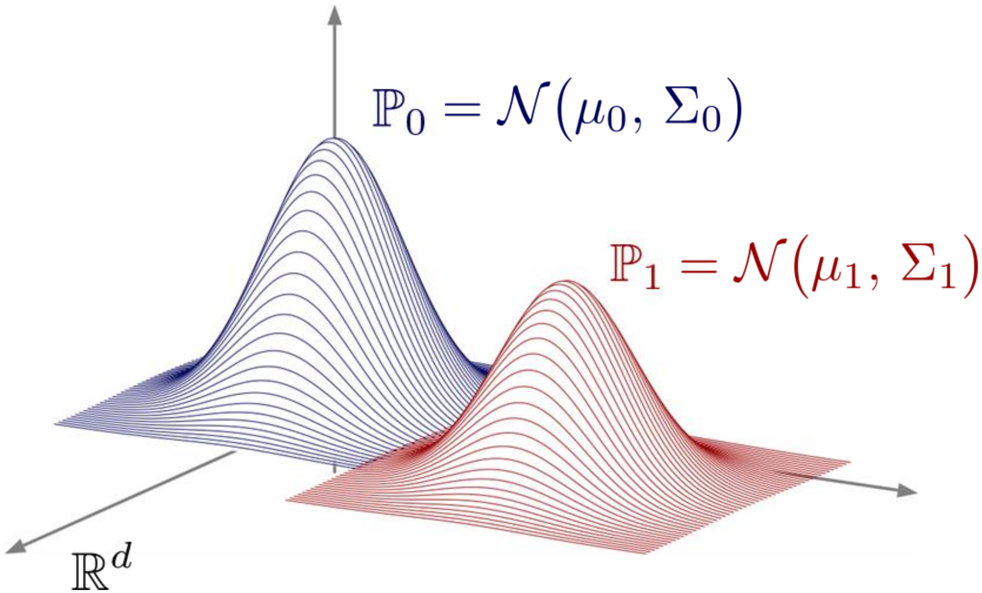

$$X \sim \mathcal{N}(\mu, \Sigma) \quad \Longrightarrow \quad f_X(x) = \frac{1}{(2\pi)^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}}} \exp\Big(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\Big)$$

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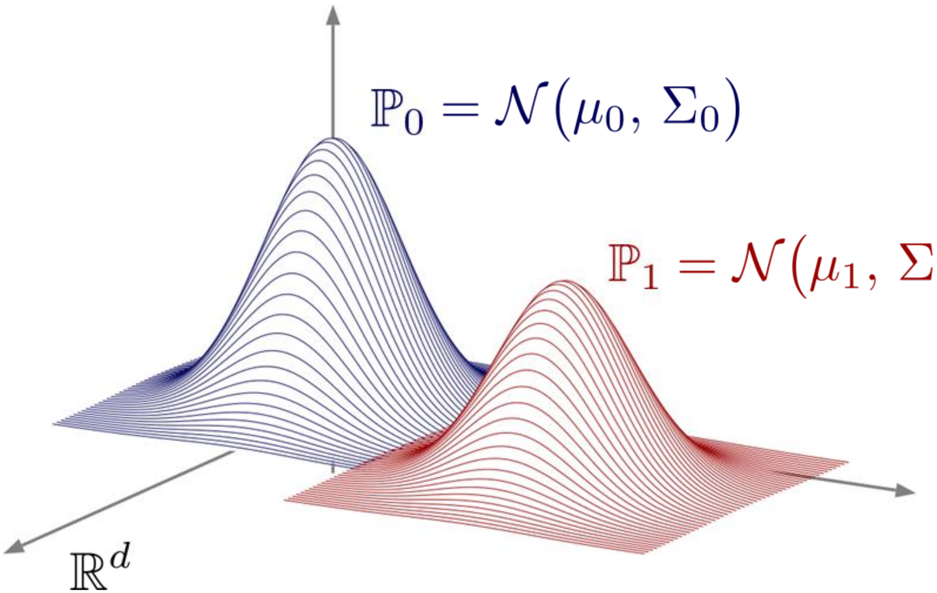

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Assume (to simplify): $\Sigma_0 = \Sigma_1 = \Sigma$

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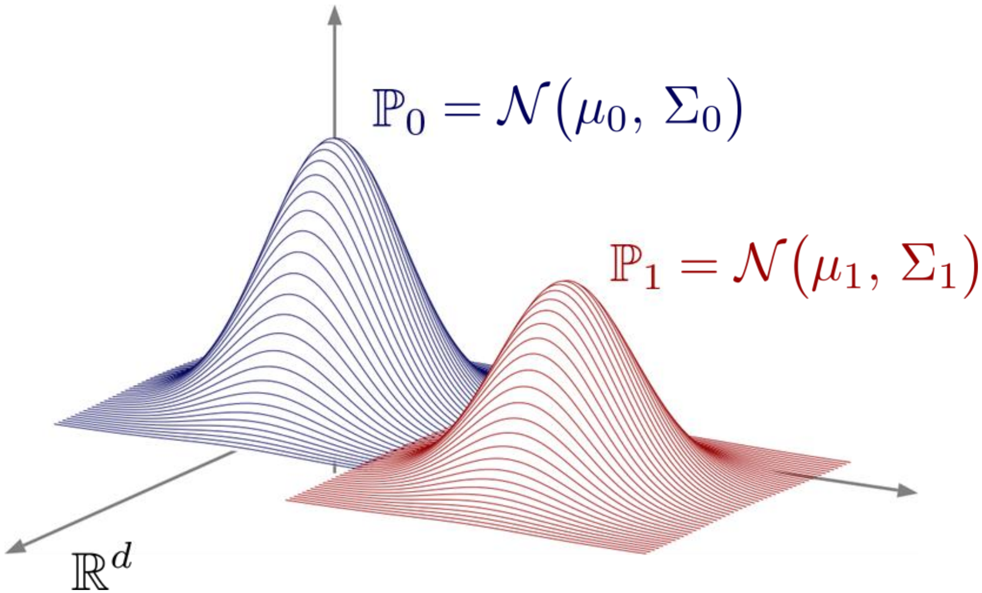

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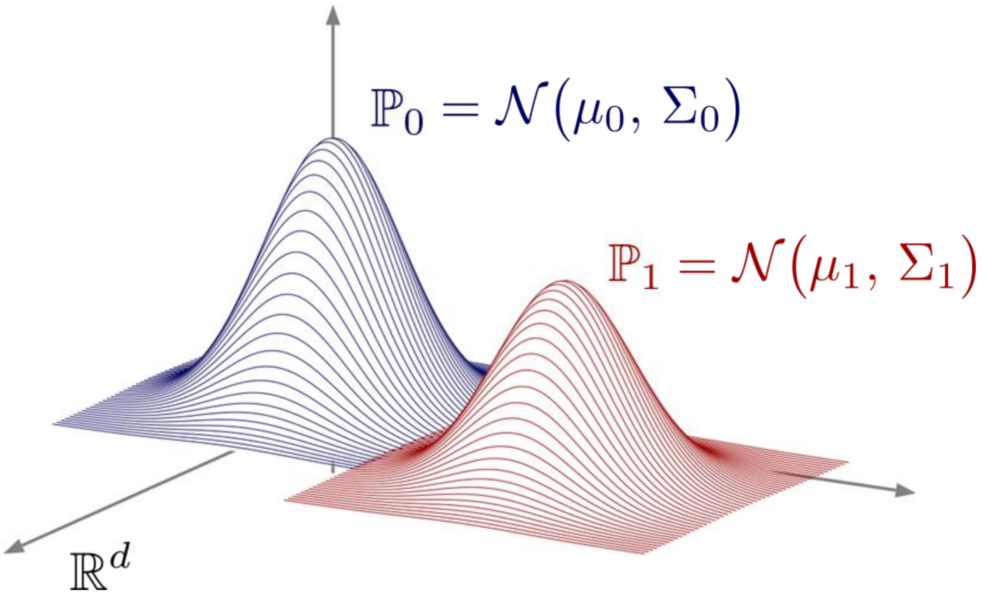

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$$L(x) > T$$

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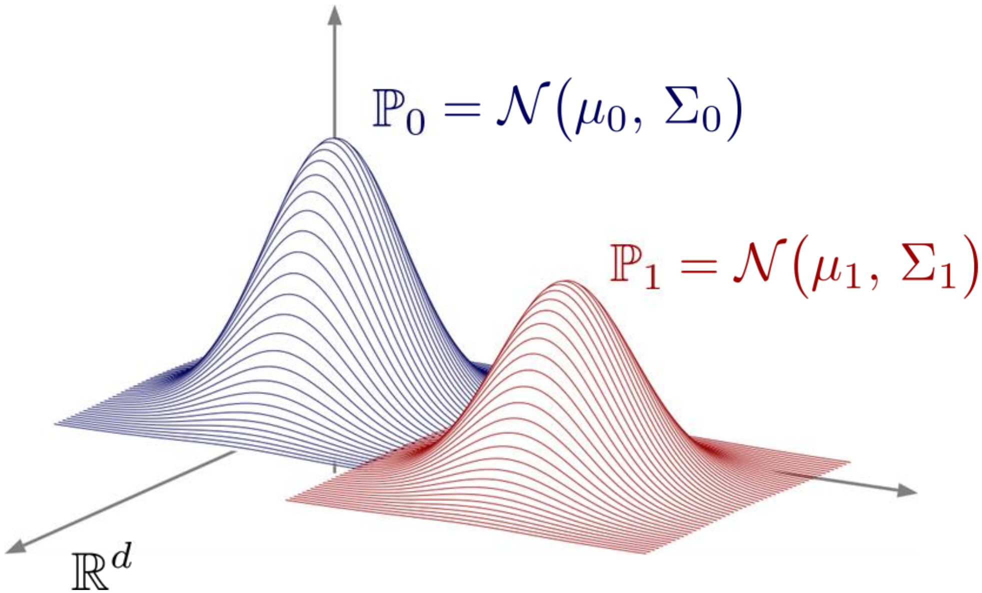

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$$L(x) > T \quad \Longleftrightarrow \quad \Psi(x) := \left(x - \frac{\mu_0 + \mu_1}{2}\right)^\top \Sigma^{-1}(\mu_1 - \mu_0) \; > \; \log T$$

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Probability of error

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$$\mathrm{Err}(\Psi) = \mathbb{P}\big(\Psi(X) > 0 \;\;\&\;\; \mathbb{P}_0 \text{ true}\big) \; + \; \mathbb{P}\big(\Psi(X) \leq 0 \;\;\&\;\; \mathbb{P}_1 \text{ true}\big)$$

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$$= \mathbb{P}\big(\Psi(X) > 0 \mid \mathbb{P}_0\big) \cdot \mathbb{P}\big(\mathbb{P}_0\big) \;+\; \mathbb{P}\big(\Psi(X) \leq 0 \mid \mathbb{P}_1\big) \cdot \mathbb{P}\big(\mathbb{P}_1\big)$$

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Fisher linear discriminant function

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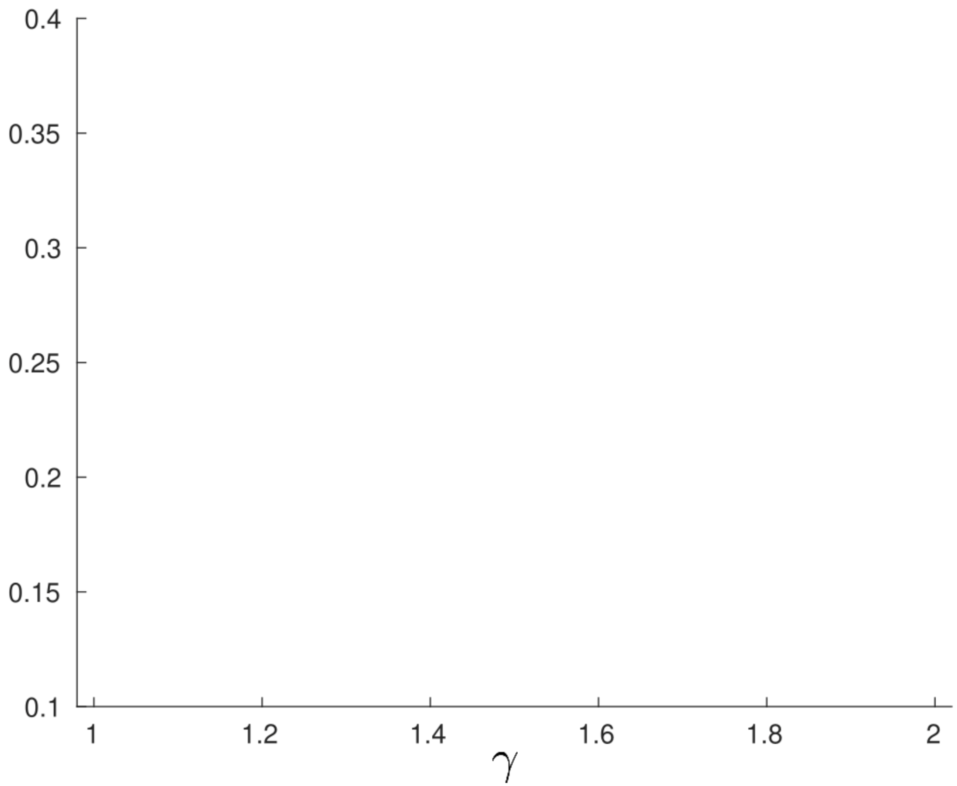

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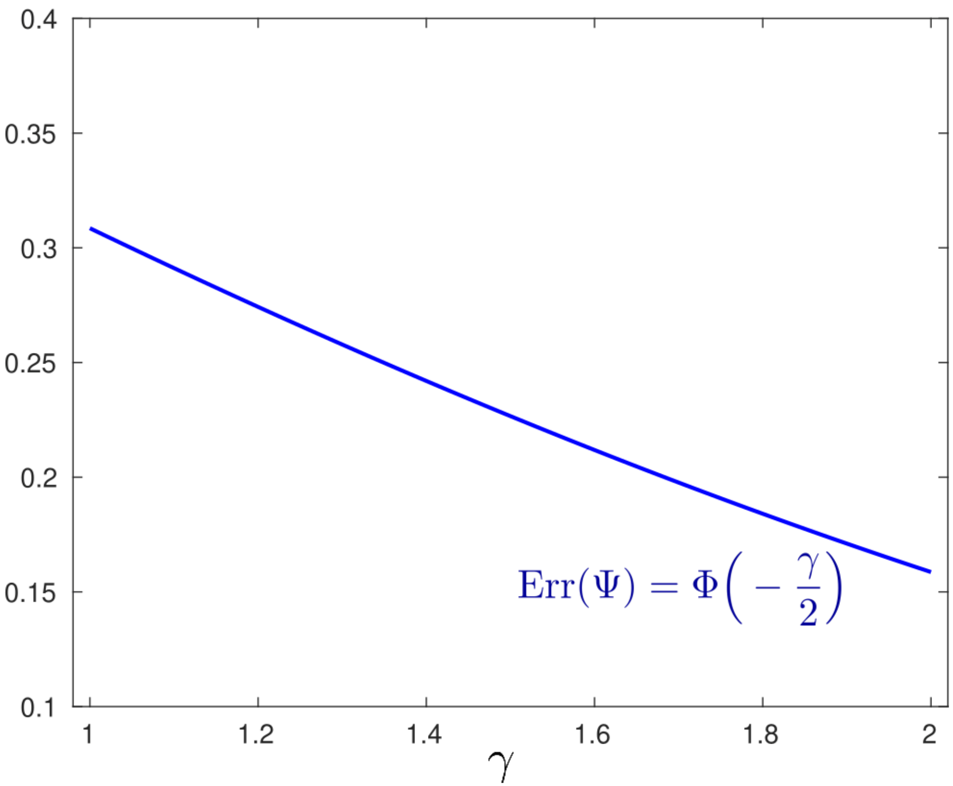

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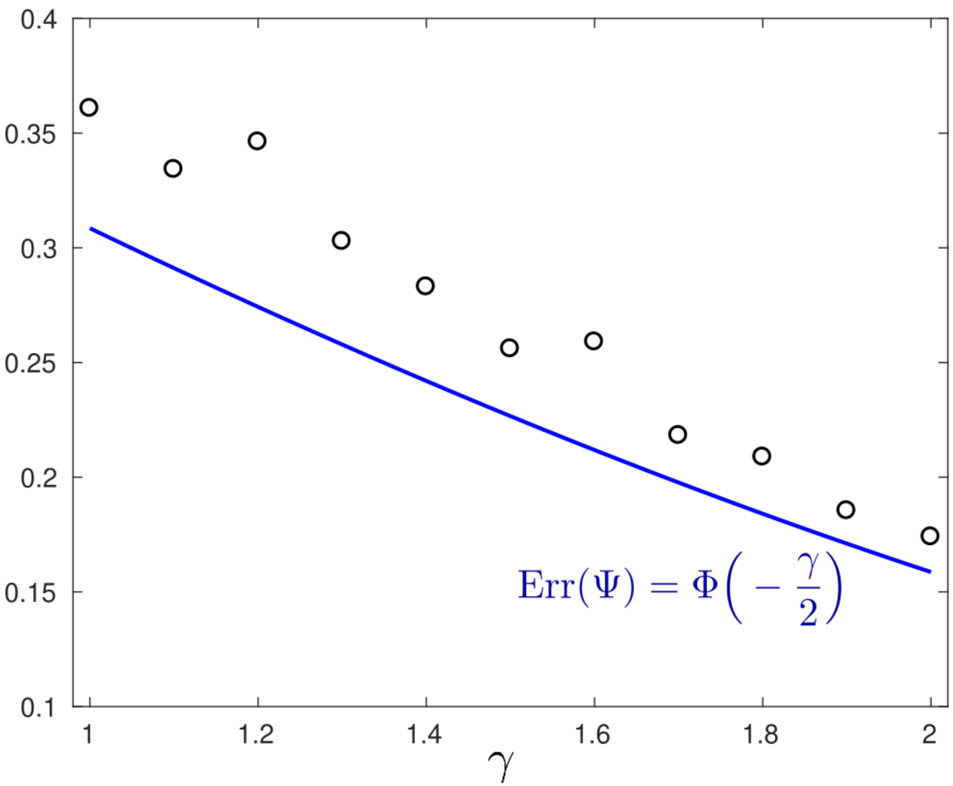

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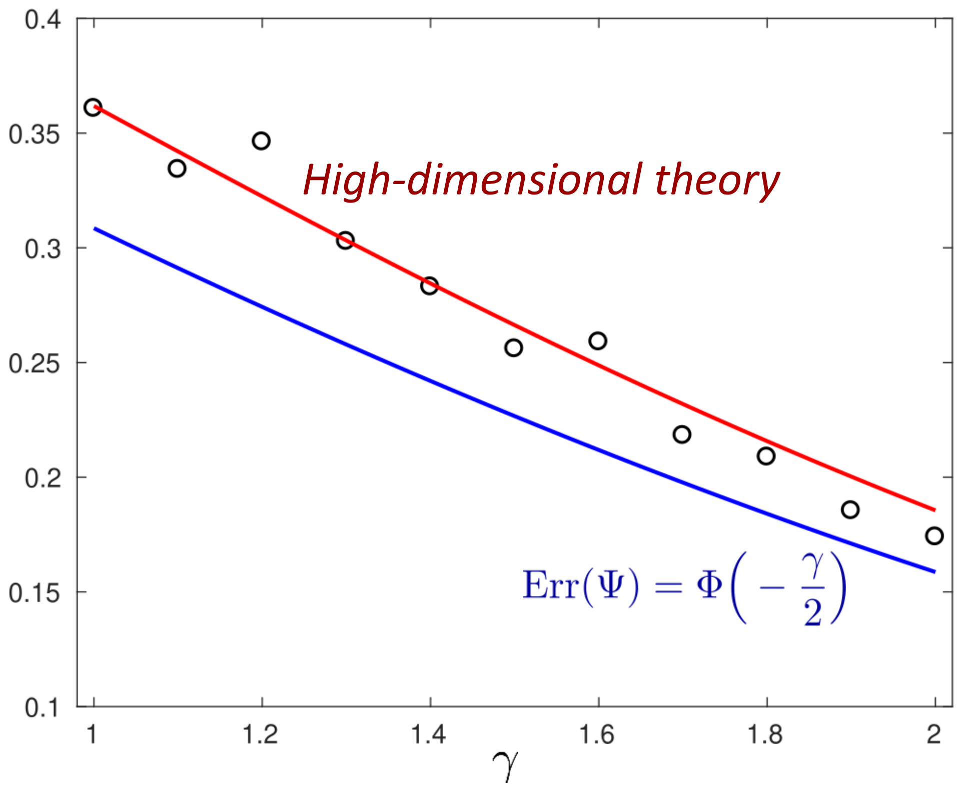

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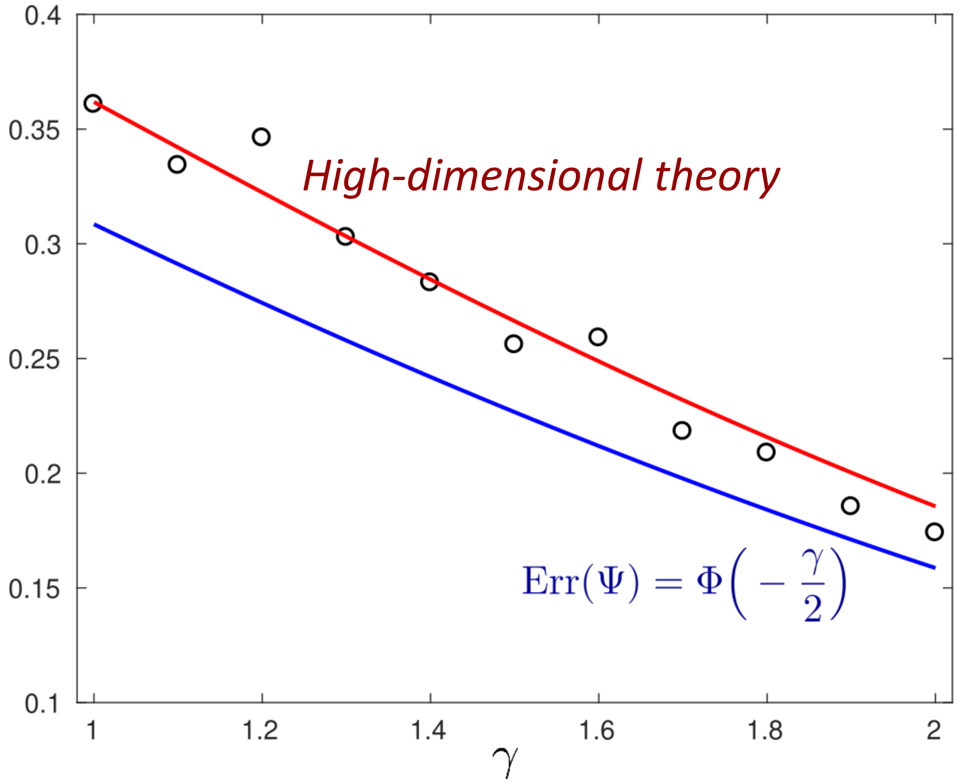

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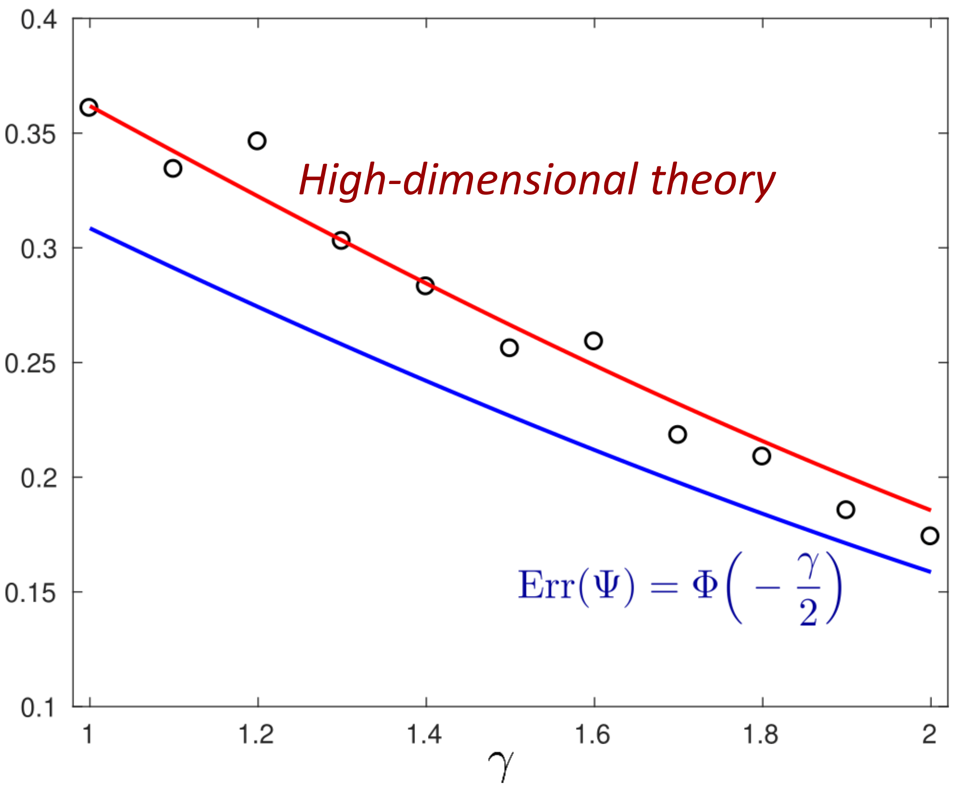

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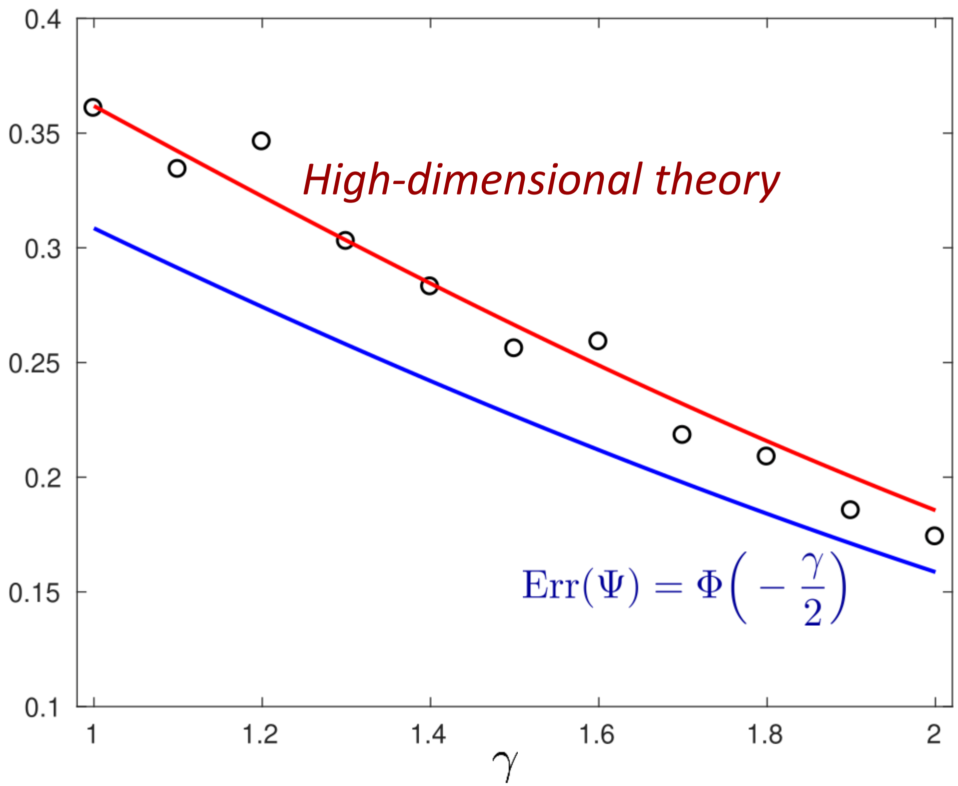

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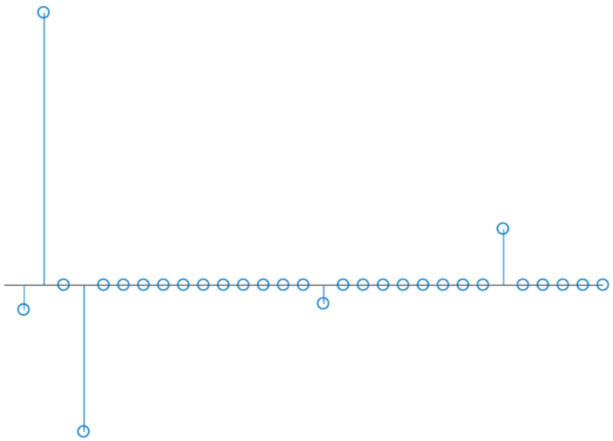

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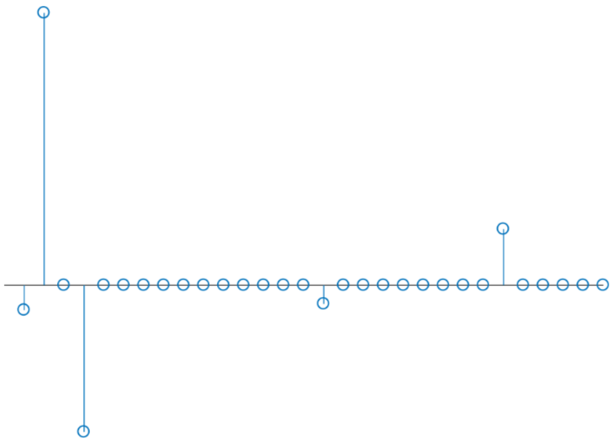

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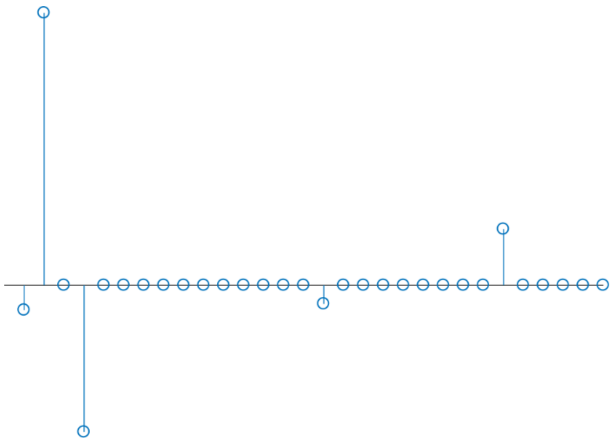

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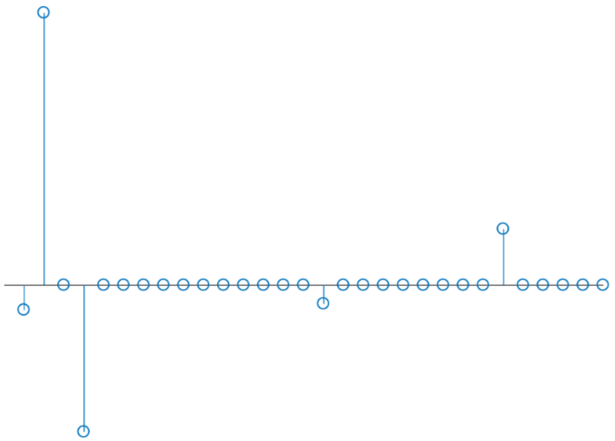

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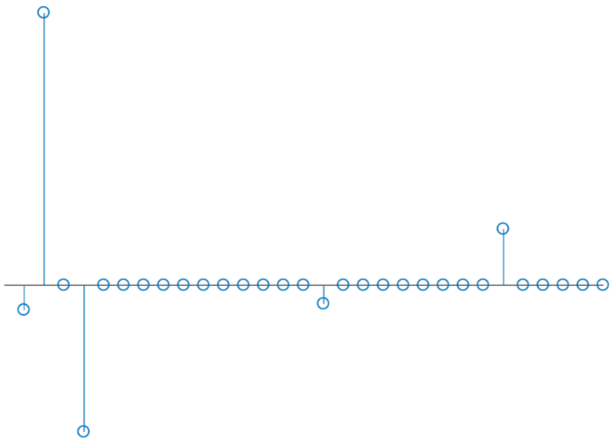

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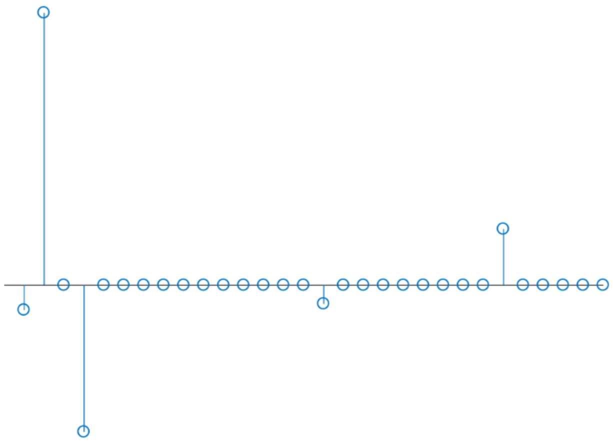

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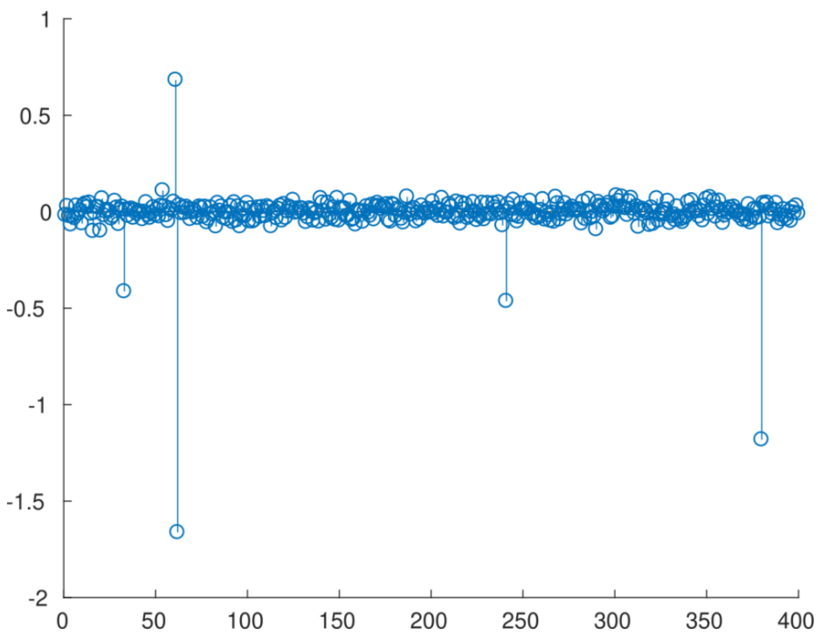
$\widehat{\boldsymbol{\mu}}'_{\mathbf{0}}$



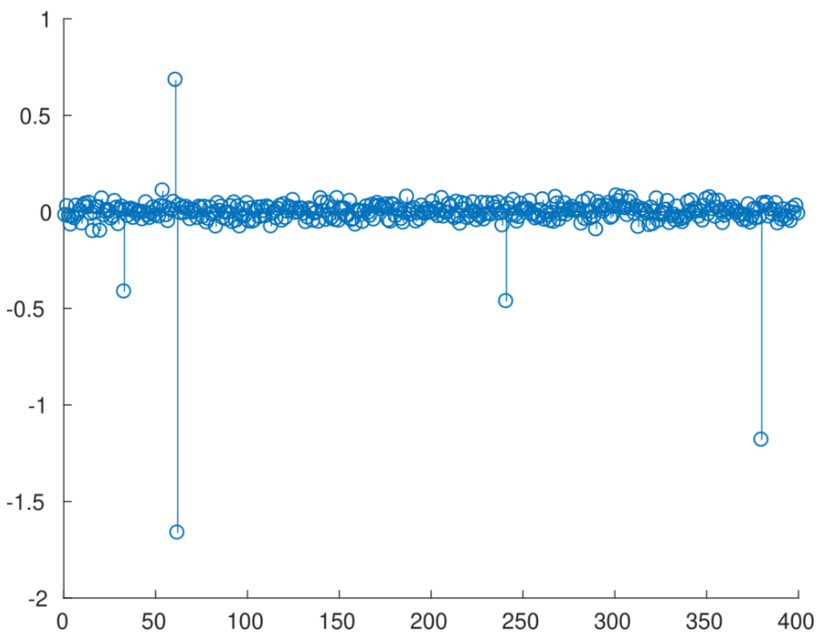
Example

hard-thresholding operator

$$H_\lambda(x) := \begin{cases} x & , \text{ if } |x| > \lambda \\ 0 & , \text{ if } |x| \leq \lambda \end{cases}$$

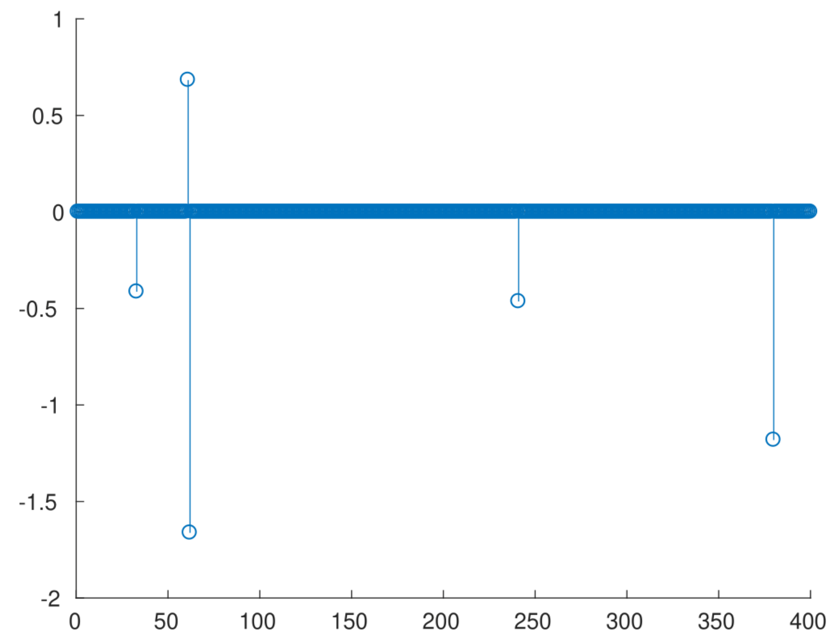
$$d = 400 \qquad n = 800 \qquad s = 5 \qquad \lambda = \sqrt{\frac{2 \log\, d}{n}} = 0.1224$$

$\widehat{\boldsymbol{\mu}}_{\mathbf{0}}'$



$\xrightarrow{H_\lambda(\cdot)}$

$\widehat{\boldsymbol{\mu}}_{\mathbf{0}}$



Same Experiments

Same Experiments

Assume $\mathbb{P}(\mathbb{P}_0) = \mathbb{P}(\mathbb{P}_1)$ $\quad \Sigma = I_d$ $\quad d = 400$ $\quad n_0 = n_1 = 800$

Vary $\gamma = \|\mu_0 - \mu_1\|_2$ between 1 and 2

Test with $\widehat{\Psi}(x)$ over 5000 random trials

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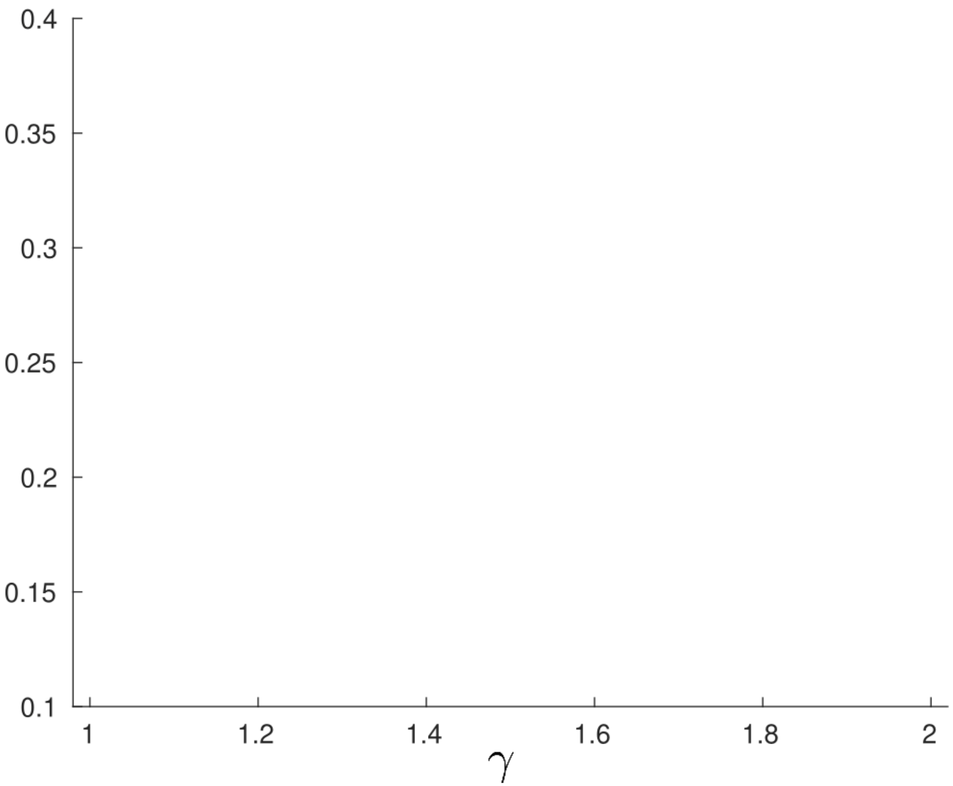
Probability of error



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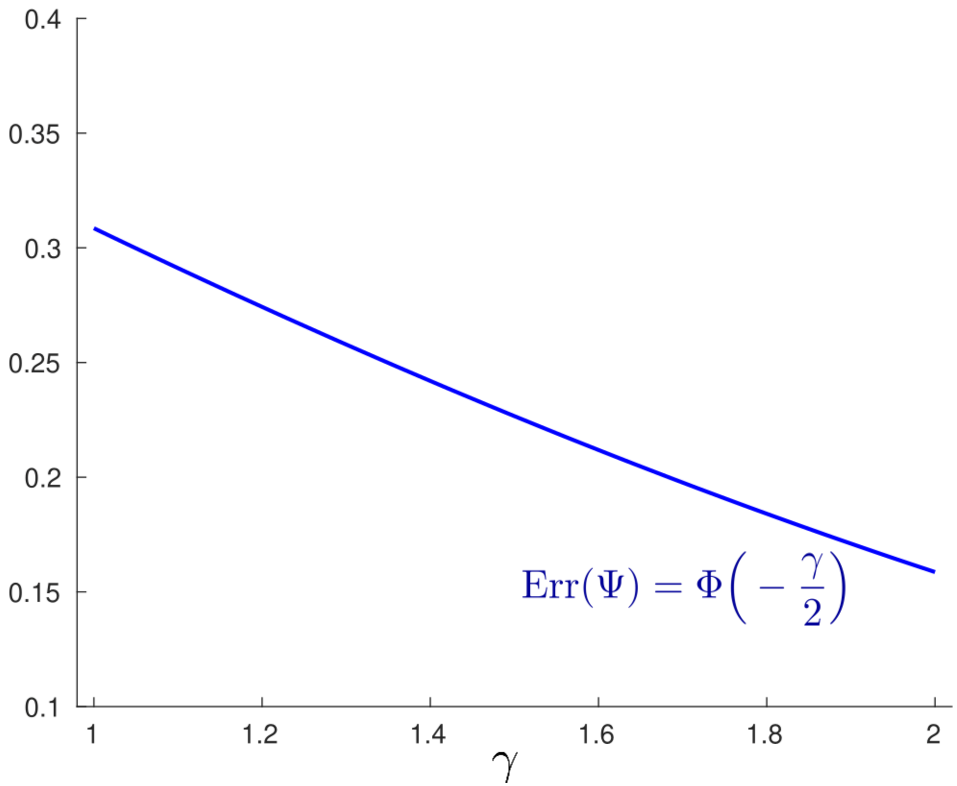

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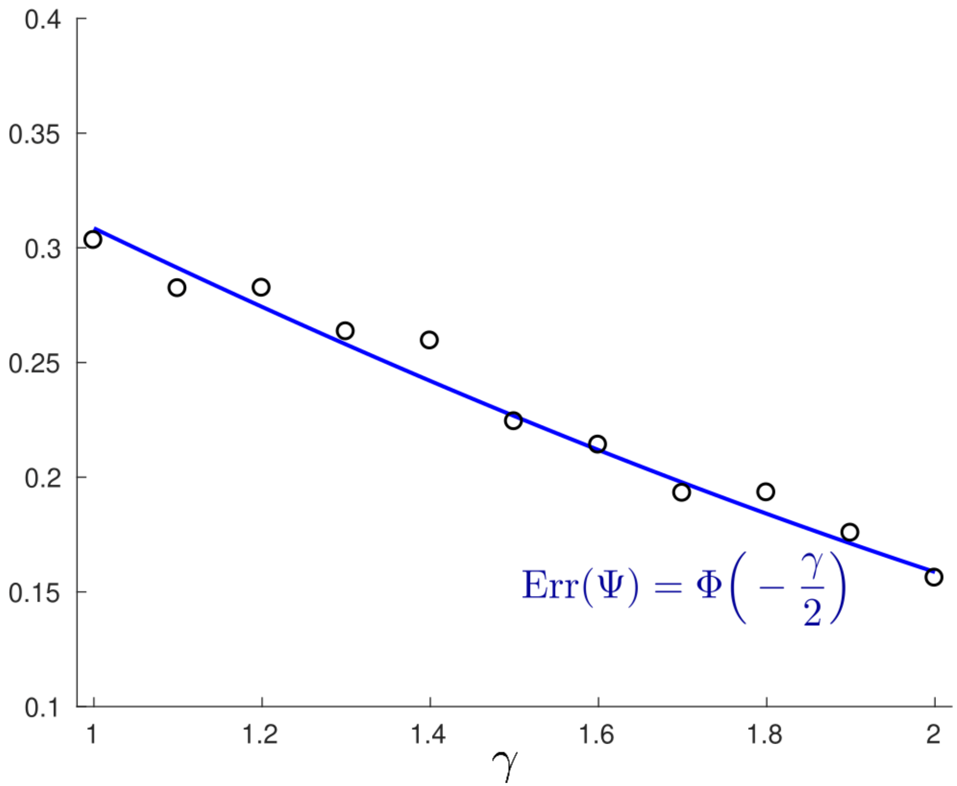

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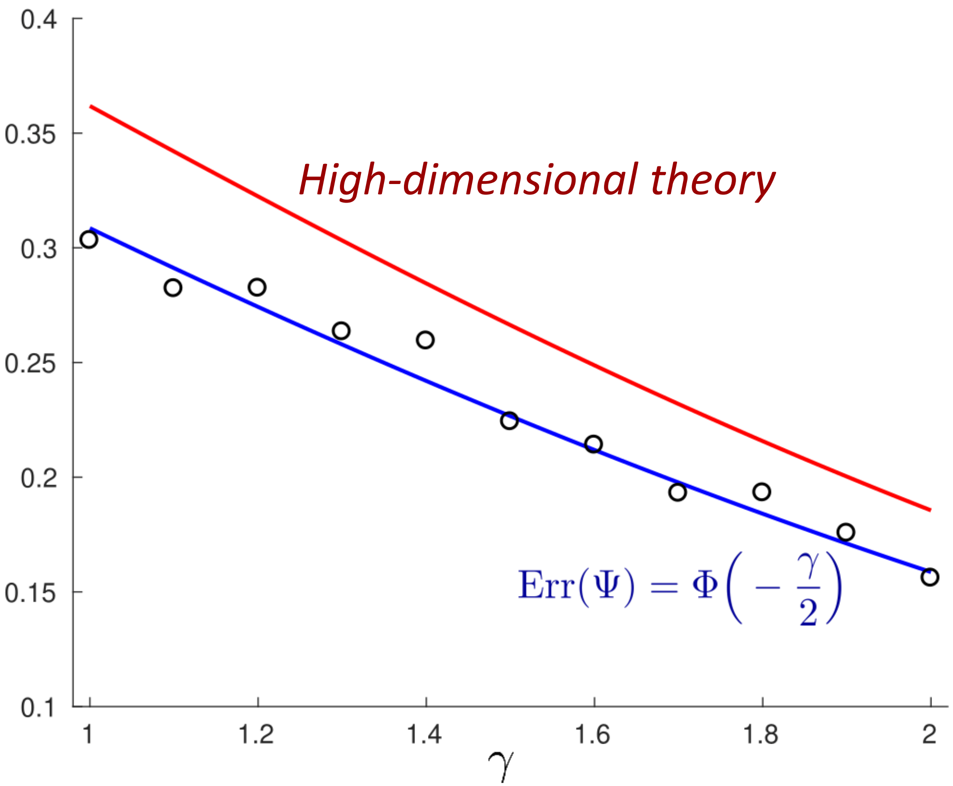

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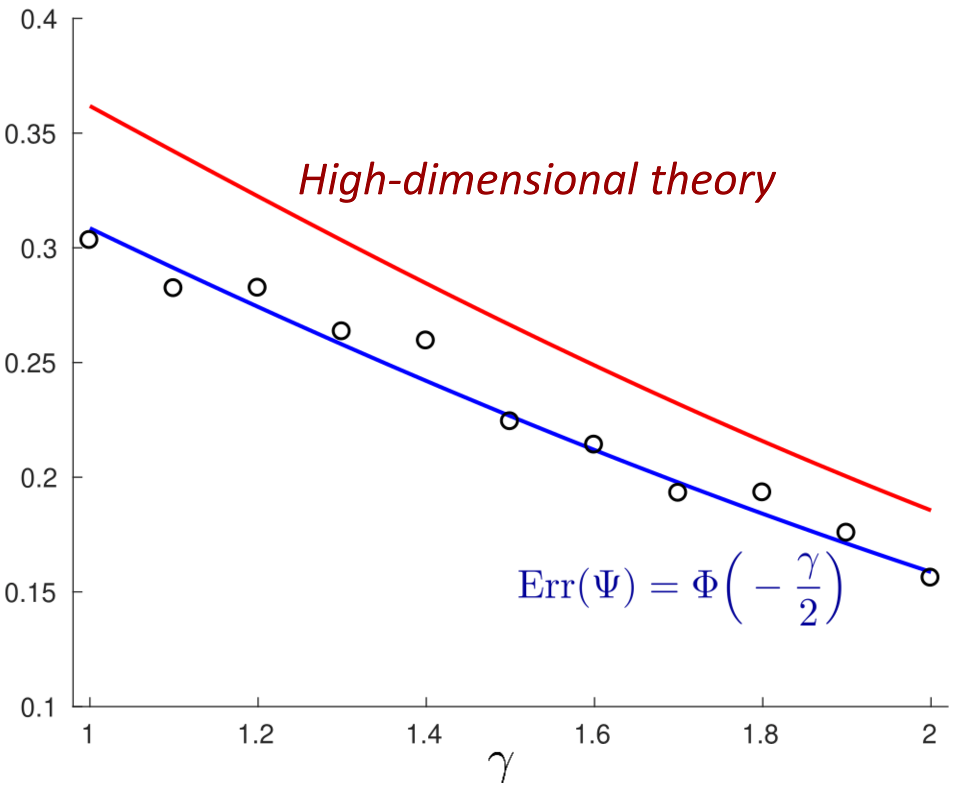

Sparsity makes problem low-dimensional

Outline

- *Motivation: Hypothesis Testing in High-Dimensions*
- Introduction to LASSO and other sparsity problems
- Gaussian graphical model selection
- Matrix completion

A Crime Problem

A Crime Problem

City	Police funding/ resident ($/ya)	% with 4 years high-school (+25)	% not in high-school (16-19)	% in college (18-24)	% with 4+ years college (25+)	Crimes/million
1	40	74	11	31	20	478
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⋮	⋮	⋮	⋮	⋮	⋮	⋮
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⋮	⋮	⋮	⋮	⋮	⋮	⋮
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Goal: Predict *# crimes / million* based on the other indicators

Linear Regression

City	Police funding per resident	% with 4 years high-school (+25)	% in high-school (16-19)	% in college (18-24)	% with 4+ years college (25+)	Crimes/million
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Linear Regression

Linear/affine model

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Linear Regression

Linear/affine model

$d = 5$ predictors

City	Police funding per resident	% with 4 years high-school (+25)	% in high-school (16-19)	% in college (18-24)	% with 4+ years college (25+)	Crimes/million
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⋮	⋮	⋮	⋮	⋮	⋮	⋮
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$\xleftrightarrow{\quad d \quad}$

Linear Regression

Linear/affine model

$d = 5$ predictors

$n = 50$ samples

n

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d

Linear Regression

Linear/affine model

$d = 5$ predictors

$n = 50$ samples

$$y_i \simeq x_0 + \sum_{j=1}^{d} a_{ij} x_j \qquad i = 1, \ldots, n$$

n

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response variable (crime rate)

n

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Linear Regression

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$$y_i \simeq x_0 + \sum_{j=1}^{d} a_{ij} x_j \qquad i = 1, \ldots, n$$

offset

response variable (crime rate)

Linear Regression

Linear/affine model

$d = 5$ predictors

$n = 50$ samples

$$y_i \simeq x_0 + \sum_{j=1}^{d} a_{ij} x_j \qquad i = 1, \dots, n$$

offset

response variable (crime rate)

table entry

n

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Linear Regression

Linear/affine model

$d = 5$ predictors

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$$y_i \simeq x_0 + \sum_{j=1}^{d} a_{ij} x_j \qquad i = 1, \ldots, n$$

offset

coefficient to be determined

response variable (crime rate)

table entry

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Linear Regression

Linear/affine model

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d

$$y_i \simeq x_0 + \sum_{j=1}^{d} a_{ij} x_j \quad i = 1, \ldots, n \quad \Longleftrightarrow \quad y \simeq x_0 1_n + Ax$$

offset

coefficient to be determined

response variable (crime rate)

table entry

Linear Regression

Linear/affine model

$d = 5$ predictors

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n

d

$$y_i \simeq x_0 + \sum_{j=1}^{d} a_{ij} x_j \quad i = 1, \ldots, n \quad \iff \quad y \simeq x_0 1_n + Ax = \begin{bmatrix} 1_n & A \end{bmatrix} \begin{bmatrix} x_0 \\ x \end{bmatrix}$$

offset

coefficient to be determined

response variable (crime rate)

table entry

Linear Regression

Linear/affine model

$d = 5$ predictors

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d

$$\boxed{y_i \simeq x_0 + \sum_{j=1}^{d} a_{ij} x_j} \quad i = 1, \ldots, n \quad \Longleftrightarrow$$

offset

coefficient to be determined

response variable (crime rate)

table entry

$$y \simeq x_0 1_n + Ax$$

$$= \underbrace{\begin{bmatrix} 1_n & A \end{bmatrix}}_{\overline{A}} \underbrace{\begin{bmatrix} x_0 \\ x \end{bmatrix}}_{\overline{x}}$$

Linear Regression

Linear/affine model

$d = 5$ predictors

$n = 50$ samples

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offset

coefficient to be determined

response variable (crime rate)

table entry

$$y \simeq x_0 1_n + Ax = \underbrace{\begin{bmatrix} 1_n & A \end{bmatrix}}_{\overline{A}} \underbrace{\begin{bmatrix} x_0 \\ x \end{bmatrix}}_{\overline{x}}$$

Find coefficients:

Linear Regression

Linear/affine model

$d = 5$ predictors

$n = 50$ samples

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$$y_i \simeq x_0 + \sum_{j=1}^{d} a_{ij} x_j \qquad i = 1, \ldots, n \qquad \Longleftrightarrow$$

offset

coefficient to be determined

response variable (crime rate)

table entry

$$y \simeq x_0 \mathbf{1}_n + Ax$$
$$= \underbrace{\begin{bmatrix} \mathbf{1}_n & A \end{bmatrix}}_{\overline{A}} \underbrace{\begin{bmatrix} x_0 \\ x \end{bmatrix}}_{\overline{x}}$$

Find coefficients: *least-squares*

$$\underset{\overline{x}}{\text{minimize}} \quad \frac{1}{2} \left\| y - \overline{A}\overline{x} \right\|_2^2$$

Linear Regression

Linear/affine model

$d = 5$ predictors

$n = 50$ samples

n

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d

$$y_i \simeq x_0 + \sum_{j=1}^{d} a_{ij} x_j \qquad i = 1, \ldots, n \qquad \Longleftrightarrow \qquad y \simeq x_0 1_n + A x = \underbrace{\begin{bmatrix} 1_n & A \end{bmatrix}}_{\overline{A}} \underbrace{\begin{bmatrix} x_0 \\ x \end{bmatrix}}_{\overline{x}}$$

offset

coefficient to be determined

response variable (crime rate)

table entry

Find coefficients: *least-squares*

$$\underset{\overline{x}}{\text{minimize}} \quad \frac{1}{2} \left\| y - \overline{A}\,\overline{x} \right\|_2^2 \qquad \Longleftrightarrow \qquad \overline{x}^\star_{\text{LS}} = \left(\overline{A}^\top \overline{A} \right)^{-1} \overline{A}\, y$$

Linear Regression

$$\underset{\overline{x}}{\text{minimize}} \quad \frac{1}{2}\left\|y - \overline{A}\overline{x}\right\|_2^2$$

$$\overline{x}^\star_{\text{LS}} = \left(\overline{A}^\top \overline{A}\right)^{-1} \overline{A}\, y$$

Linear Regression

$$\underset{\overline{x}}{\text{minimize}} \quad \frac{1}{2}\left\| y - \overline{A}\,\overline{x} \right\|_2^2$$

$$\overline{x}^{\star}_{\mathrm{LS}} = \left(\overline{A}^{\top}\overline{A}\right)^{-1}\overline{A}\,y$$

$$= \begin{bmatrix} 489.6486 \\ 10.9807 \\ -6.0885 \\ 5.4803 \\ 0.3770 \\ 5.5005 \end{bmatrix}$$

Linear Regression

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$$\overline{x}^\star_{\text{LS}} = \left(\overline{A}^\top\overline{A}\right)^{-1}\overline{A}\,y$$

$$= \begin{array}{cl} \left[\begin{array}{r} 489.6486 \\ 10.9807 \\ -6.0885 \\ 5.4803 \\ 0.3770 \\ 5.5005 \end{array}\right] & \begin{array}{l} \textit{offset} \\ \\ \\ \\ \\ \\ \end{array} \end{array}$$

Linear Regression

$$\underset{\overline{x}}{\text{minimize}} \quad \frac{1}{2}\left\|y - \overline{A}\overline{x}\right\|_2^2$$

$$\overline{x}^\star_{\text{LS}} = \left(\overline{A}^\top \overline{A}\right)^{-1} \overline{A}\, y$$

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Linear Regression

$$\underset{\overline{x}}{\text{minimize}} \quad \frac{1}{2}\left\| y - \overline{A}\,\overline{x} \right\|_2^2$$

$$\overline{x}^{\star}_{\text{LS}} = \left(\overline{A}^{\top}\overline{A}\right)^{-1}\overline{A}\,y$$

$$= \begin{bmatrix} 489.6486 \\ 10.9807 \\ -6.0885 \\ 5.4803 \\ 0.3770 \\ 5.5005 \end{bmatrix} \begin{array}{l} \textit{offset} \\ \textit{police funding / resident (\$/year)} \\ \textit{\% of 25+ year-olds with 4+ years of high-school} \\ \\ \\ \\ \end{array}$$

Linear Regression

$$\underset{\overline{x}}{\text{minimize}} \quad \frac{1}{2}\left\| y - \overline{A}\overline{x} \right\|_2^2$$

$$\overline{x}^\star_{\text{LS}} = \left(\overline{A}^\top \overline{A}\right)^{-1} \overline{A}\, y$$

$$= \begin{array}{ll} \begin{bmatrix} 489.6486 \\ 10.9807 \\ -6.0885 \\ 5.4803 \\ 0.3770 \\ 5.5005 \end{bmatrix} & \begin{array}{l} \textit{offset} \\ \textit{police funding / resident (\$/year)} \\ \textit{\% of 25+ year-olds with 4+ years of high-school} \\ \textit{\% of 16-19 year-olds not in high-school} \\ \textit{\% of 18-24 year-olds in college} \\ \textit{\% of 25+ year-olds with 4+ years of college} \end{array} \end{array}$$

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Problems with least-squares

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little interpretability

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small bias, large variance

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Problems with least-squares

little interpretability — all coefficients contribute to prediction

small bias, large variance — zeroing coefficients can improve mean-squared error

LASSO

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least absolute selection and shrikange operator

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$$\underset{x}{\text{minimize}} \quad \frac{1}{2n}\|y - Ax\|_2^2 + \lambda\|x\|_1$$

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regularization parameter ≥ 0

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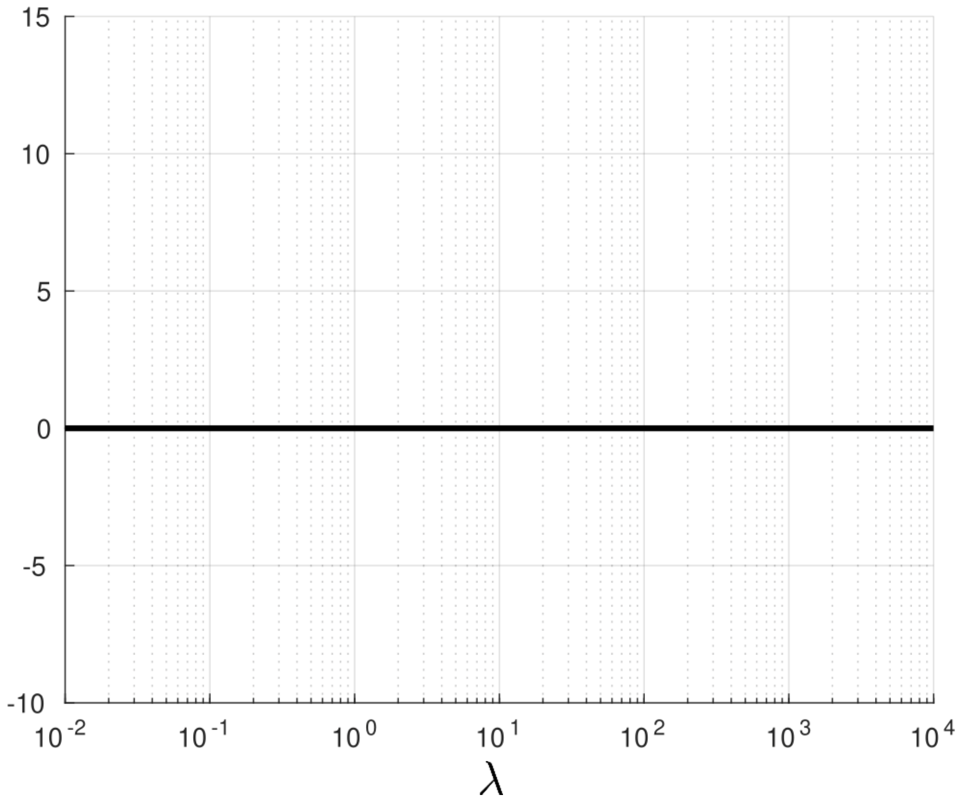
Coefficient value



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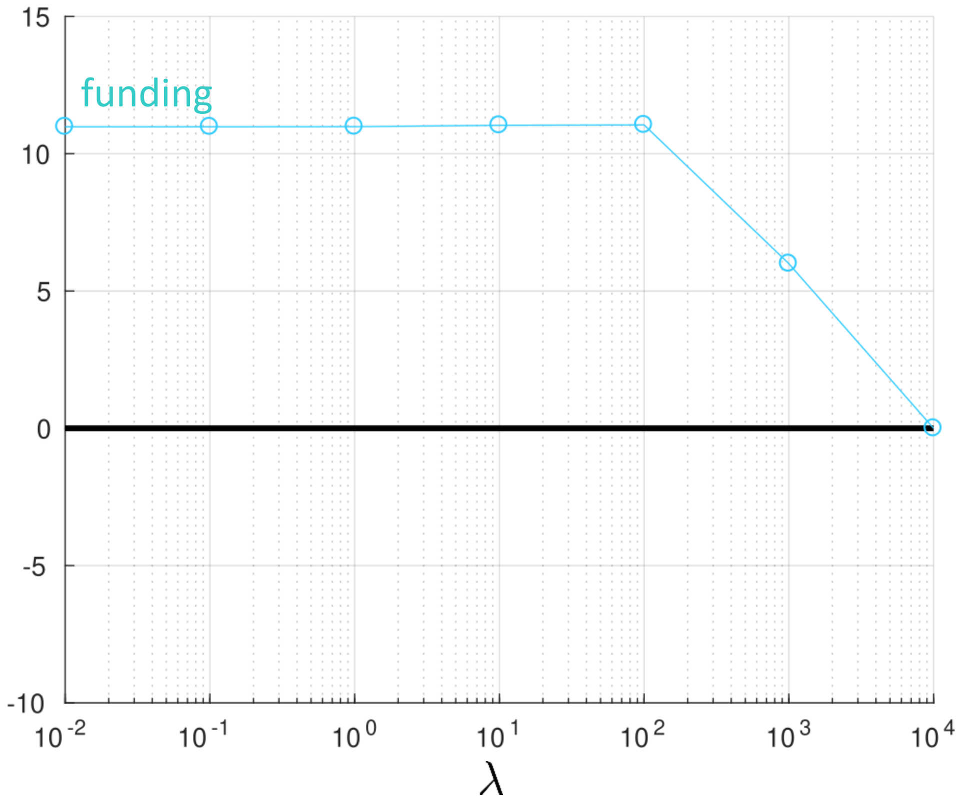

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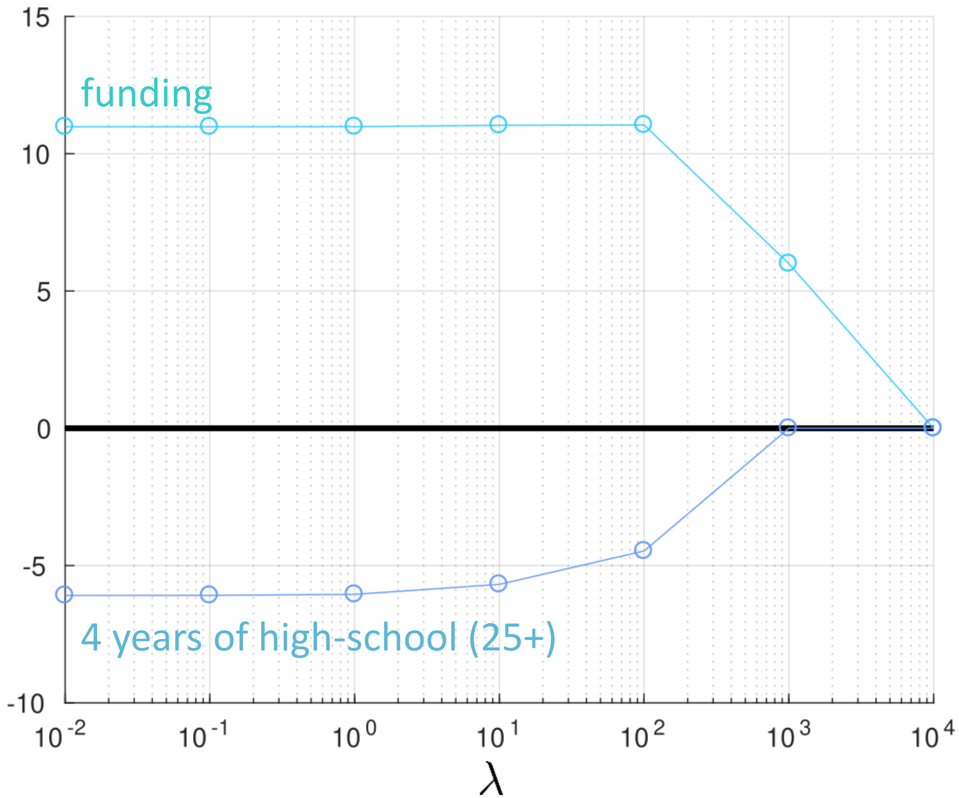

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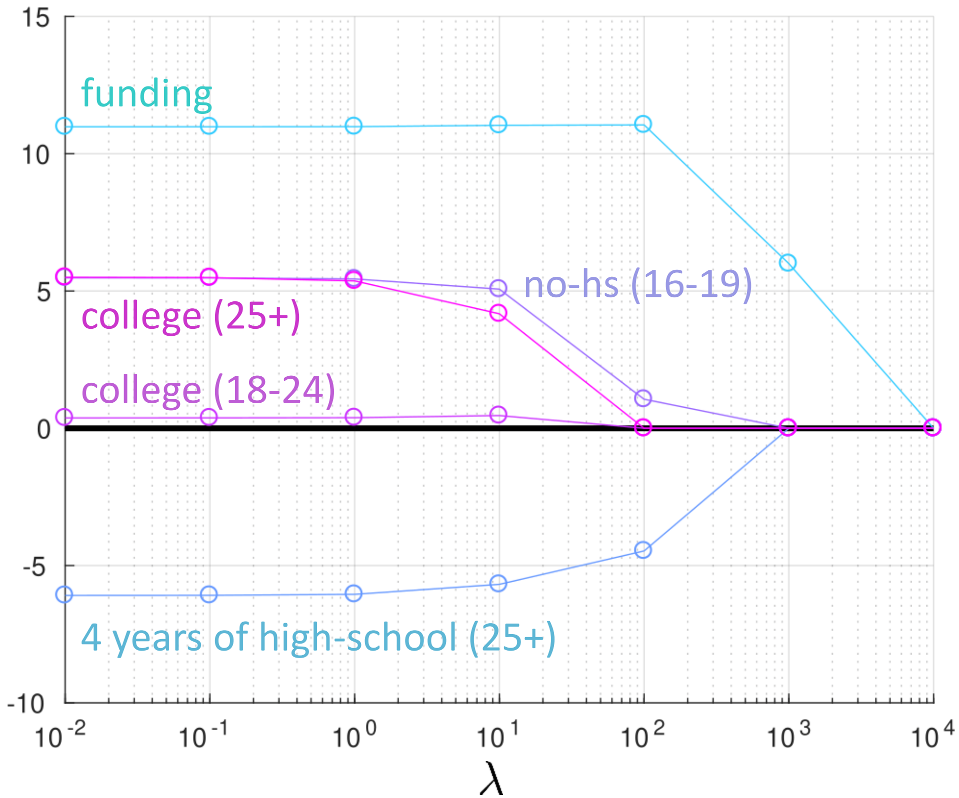

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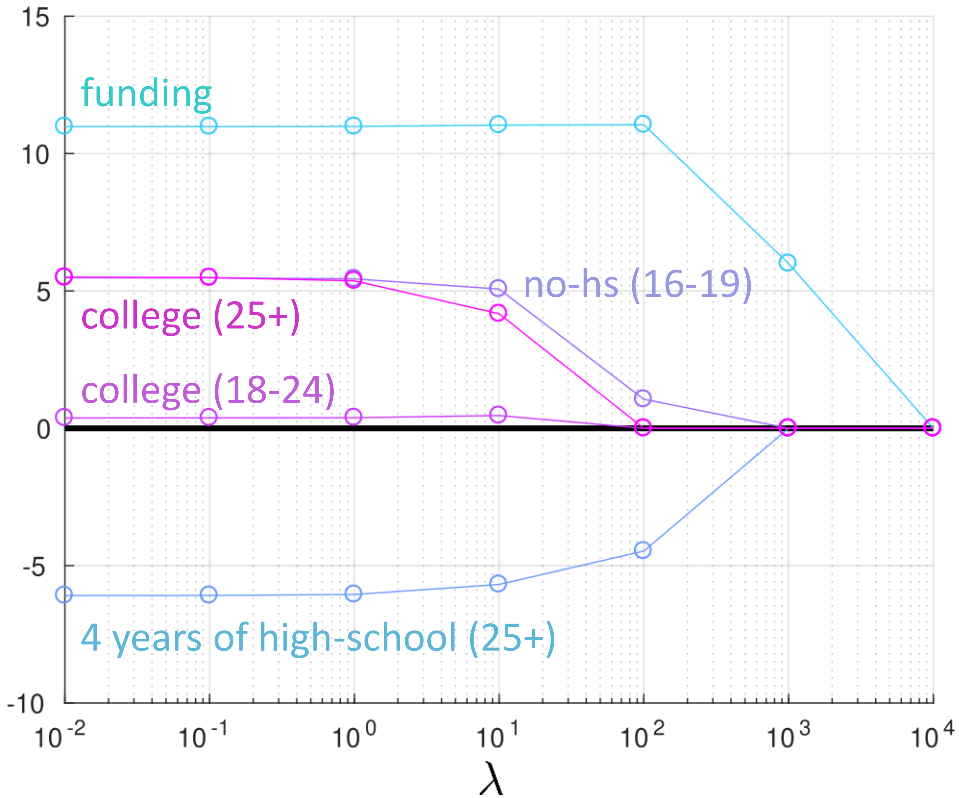

In reality, we solved ...

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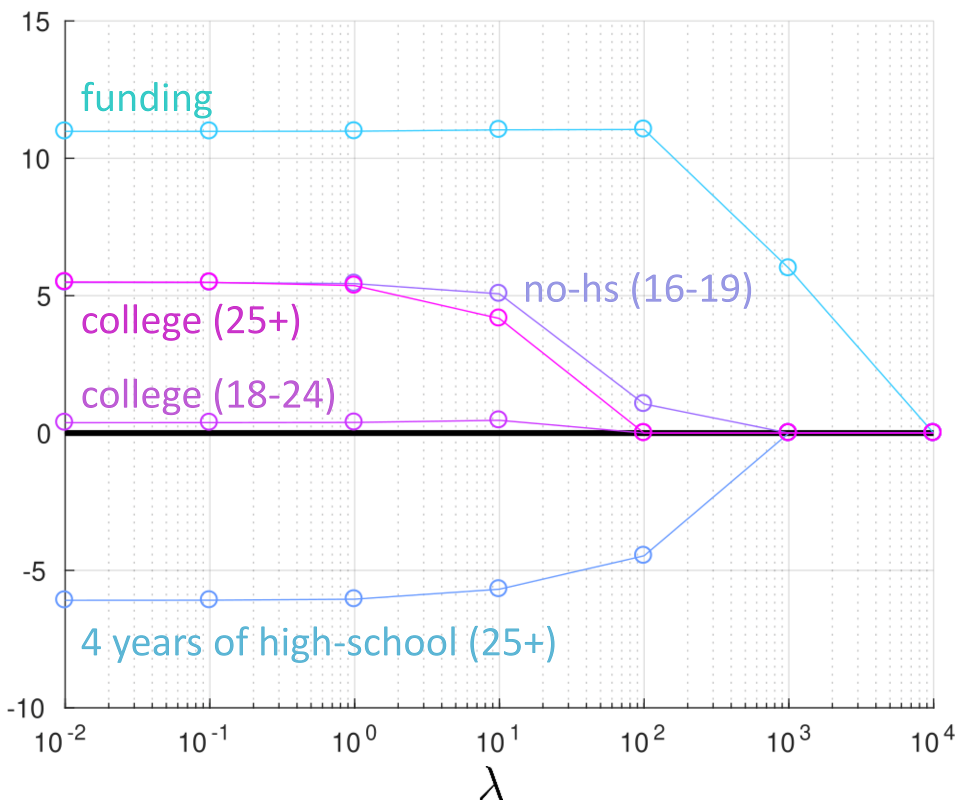

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$$\min_{x_0,\, x} \quad \frac{1}{2n}\|y - x_0 1_n - Ax\|_2^2 + \lambda\|x\|_1$$

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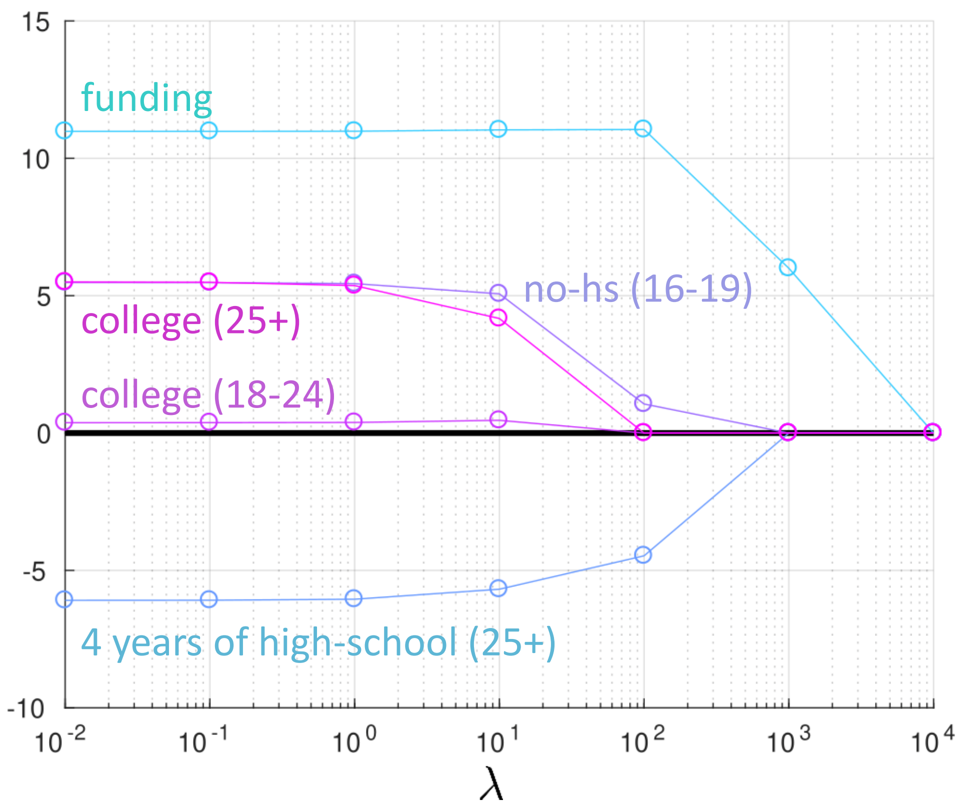

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$$\min_{x_0,\, x} \quad \frac{1}{2n}\|y - x_0 1_n - Ax\|_2^2 + \lambda\|x\|_1$$

necessary because $\frac{1}{n}\sum_{i=1}^{n} y_i \neq 0$

L1-Norm Induces Sparsity

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LASSO (aka Basis Pursuit Denoising)

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$\Updownarrow$ for some τ depending on λ

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$$\begin{aligned} &\underset{x}{\text{minimize}} && \|y - Ax\|_2^2 \\ &\text{subject to} && \|x\|_1 \le \tau \end{aligned}$$

Constrained LASSO

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Relaxed Basis Pursuit

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Relaxed Basis Pursuit

Basis Pursuit when $\sigma = 0$

L1-Norm Induces Sparsity

$$\begin{aligned} \widehat{x} \in \quad & \arg\min_{x} \quad \|x\|_1 \\ & \text{s.t.} \qquad \left\|y - Ax\right\|_2 \leq \sigma \end{aligned}$$

L1-Norm Induces Sparsity

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Assume $\sigma = 0$:

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$$\left.\right|_{y = A\widetilde{x}}$$

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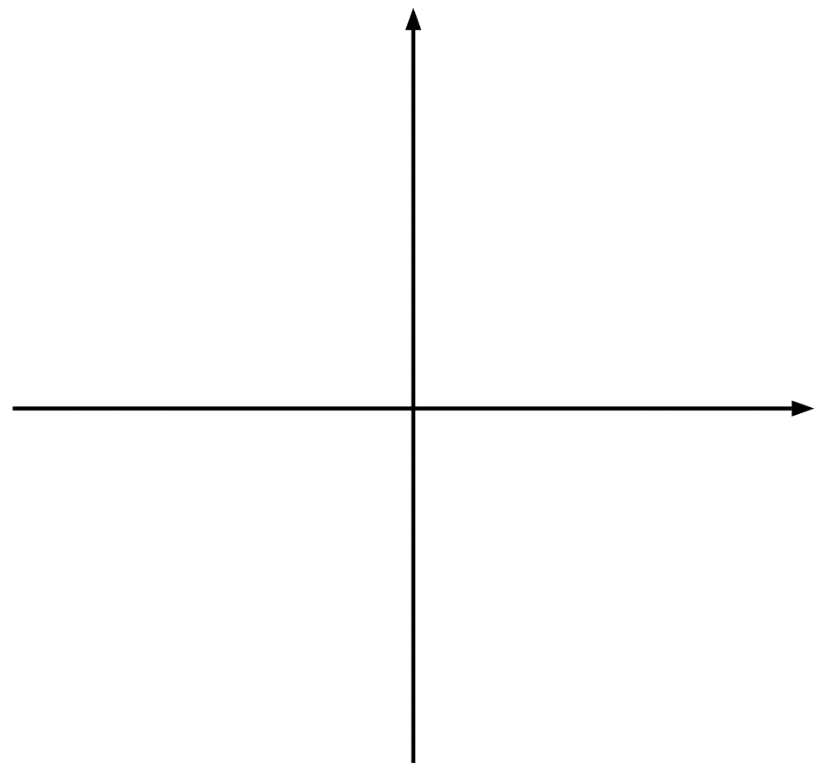

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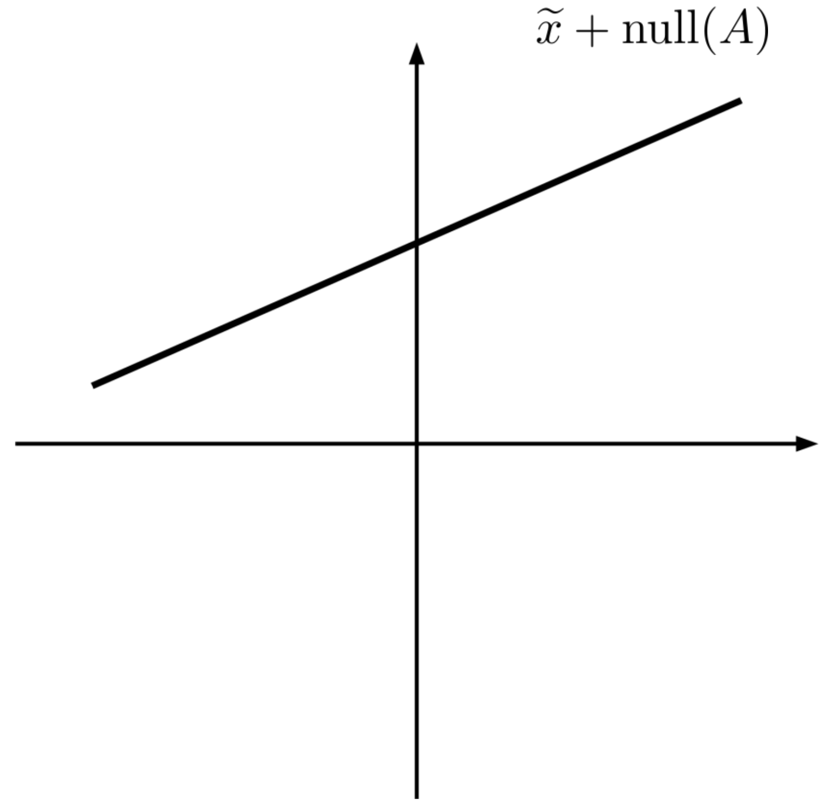

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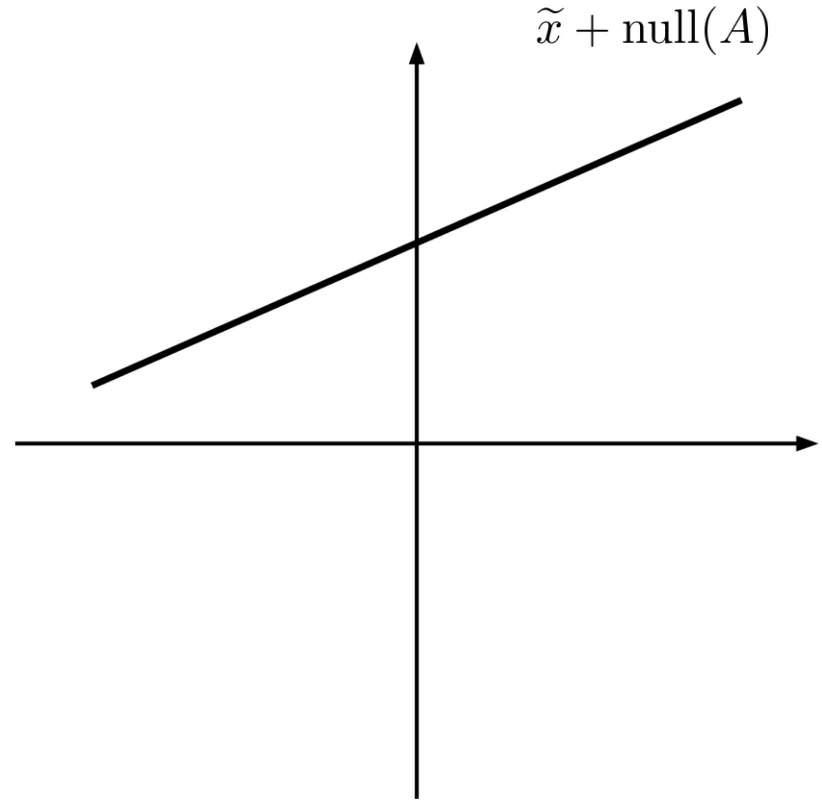

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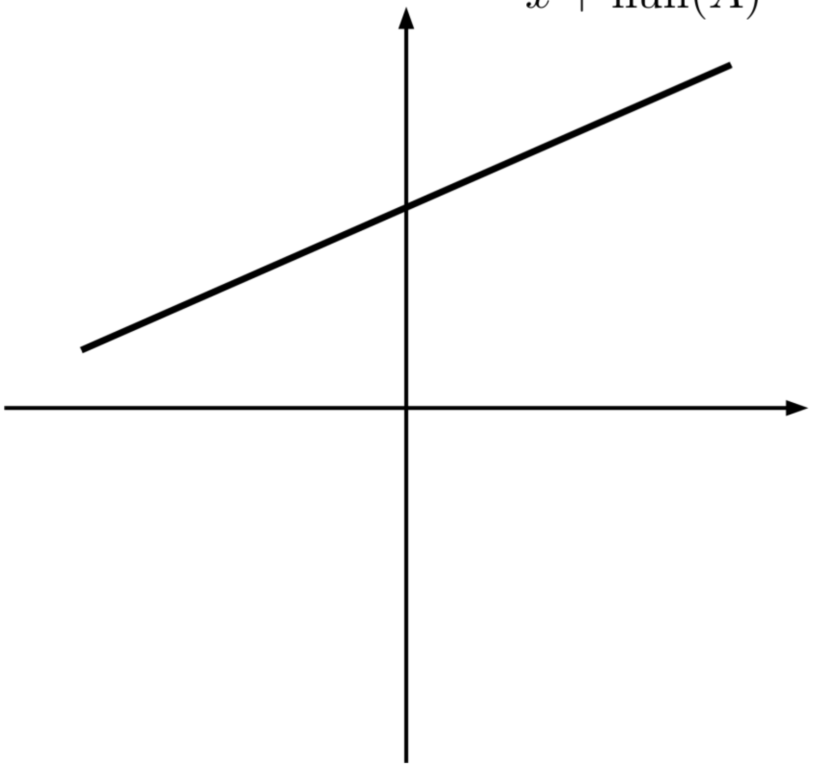
Assume $\sigma > 0$: *margin around* $\widetilde{x} + \text{null}(A)$



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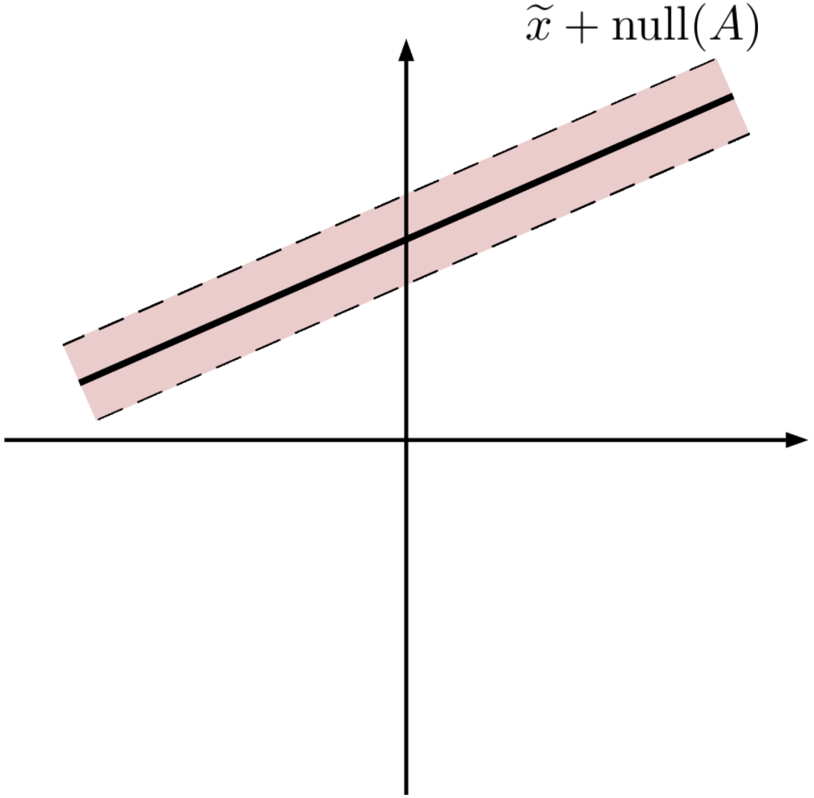

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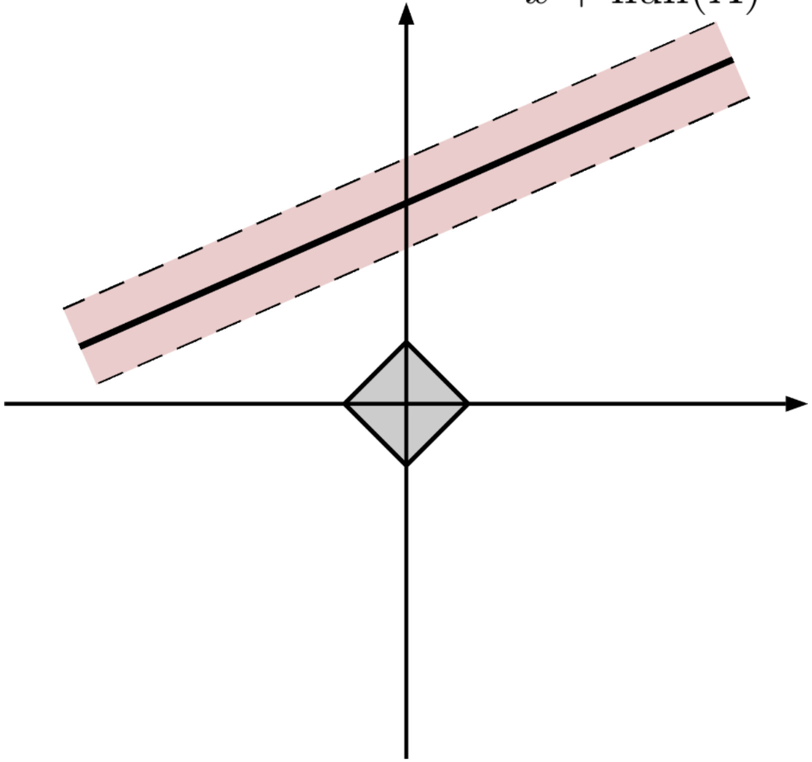

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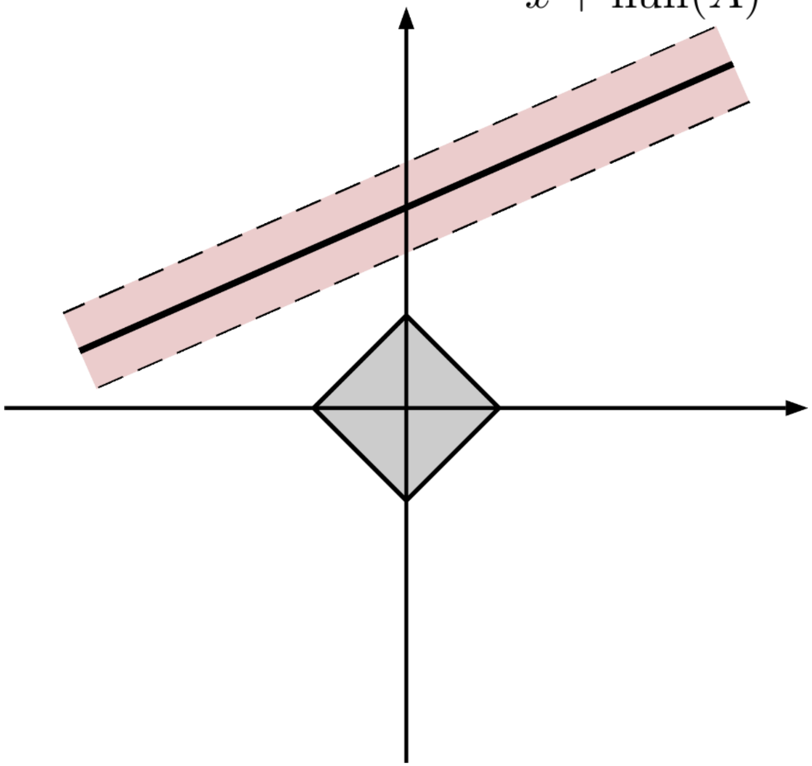

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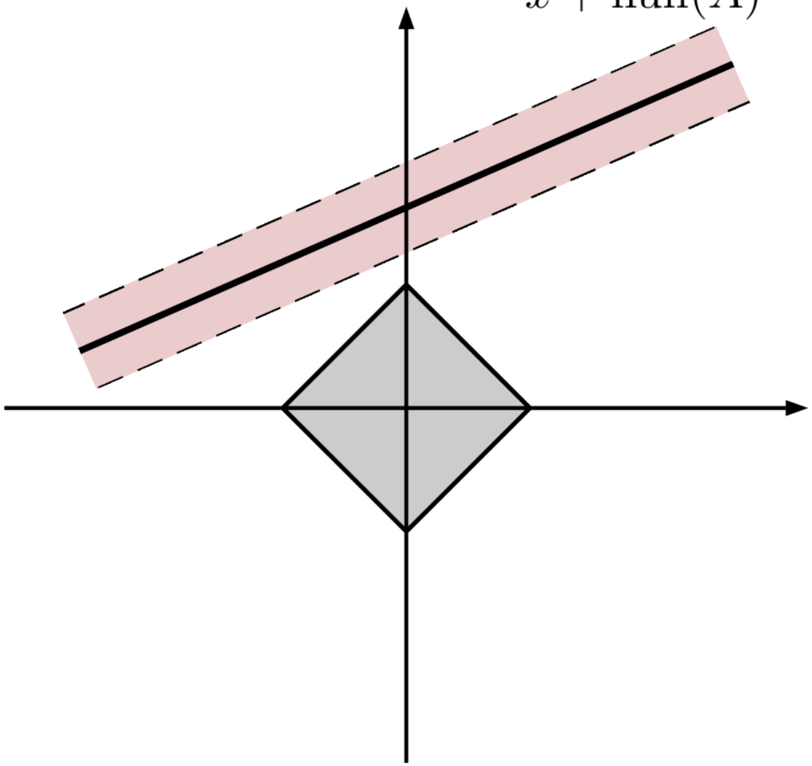

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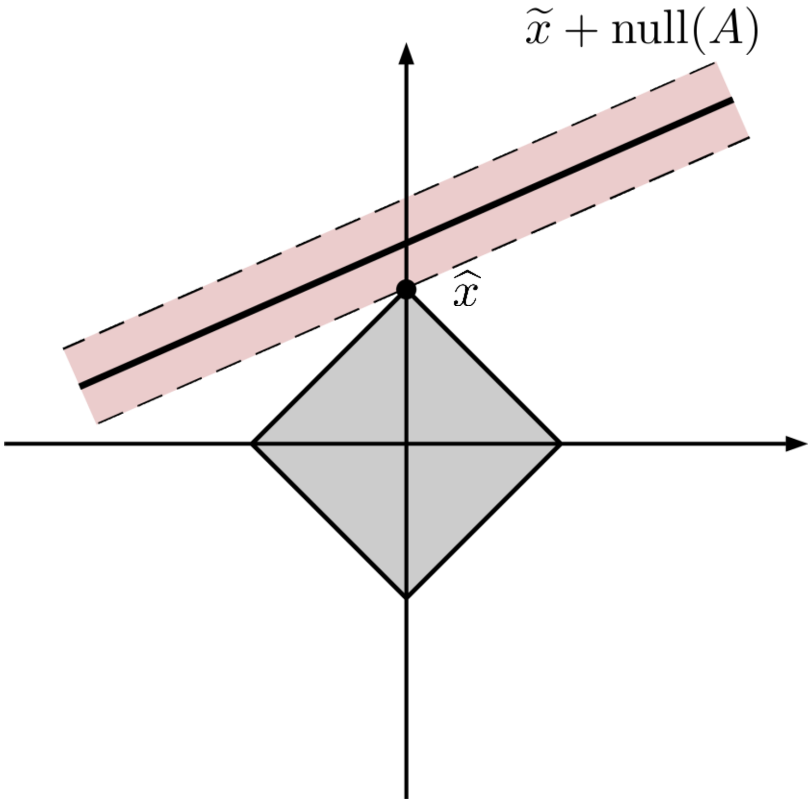

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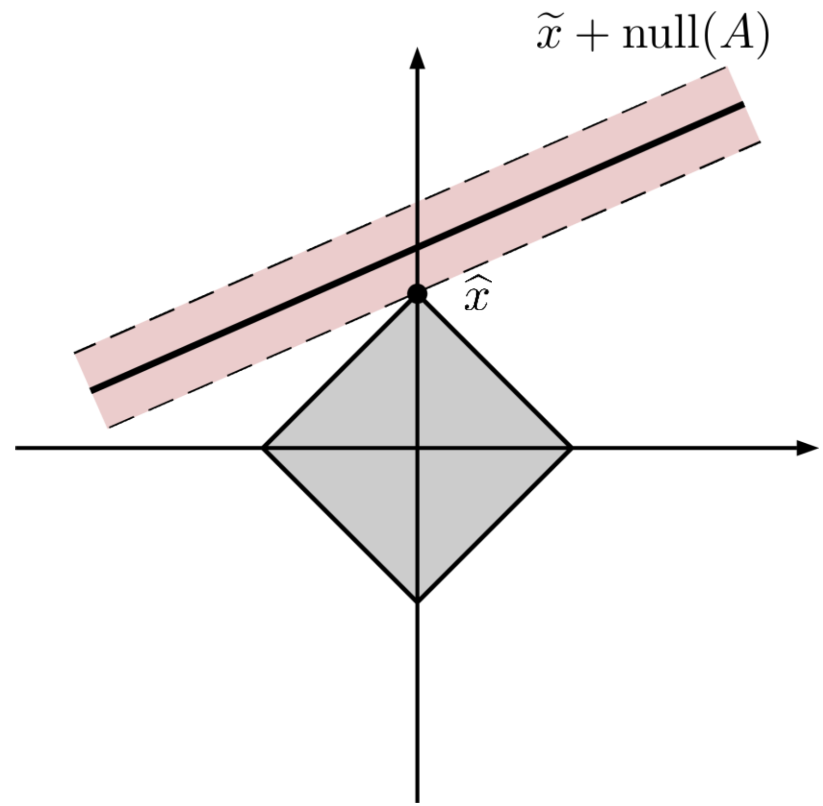

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What about the L2-norm?

$$\widehat{x}_2 \in \arg\min_{x} \|x\|_2^2 \quad \text{s.t.} \quad \|y - Ax\|_2 \leq \sigma$$

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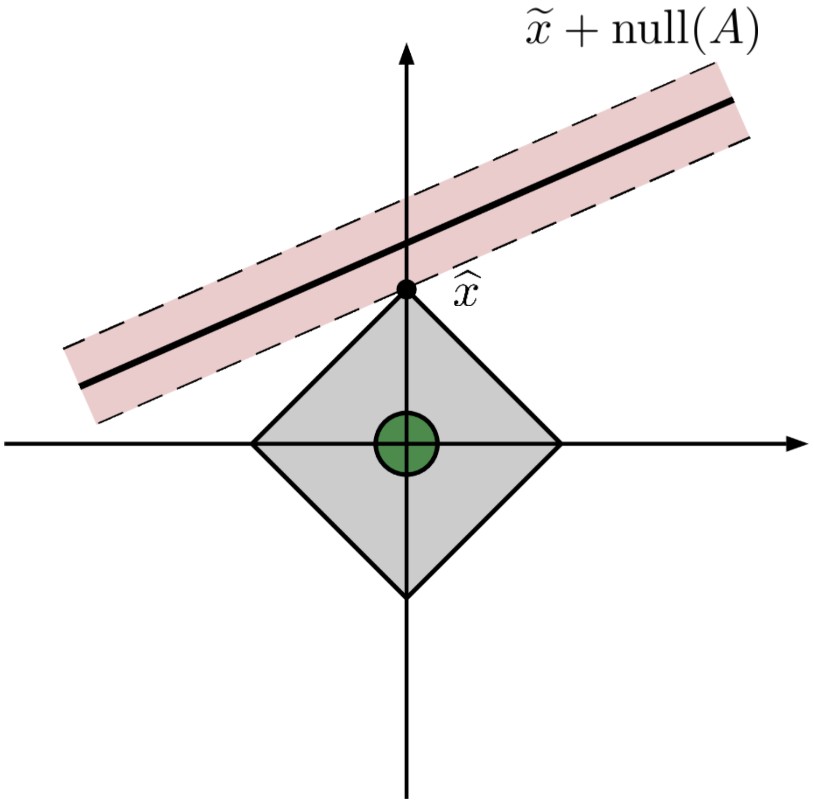

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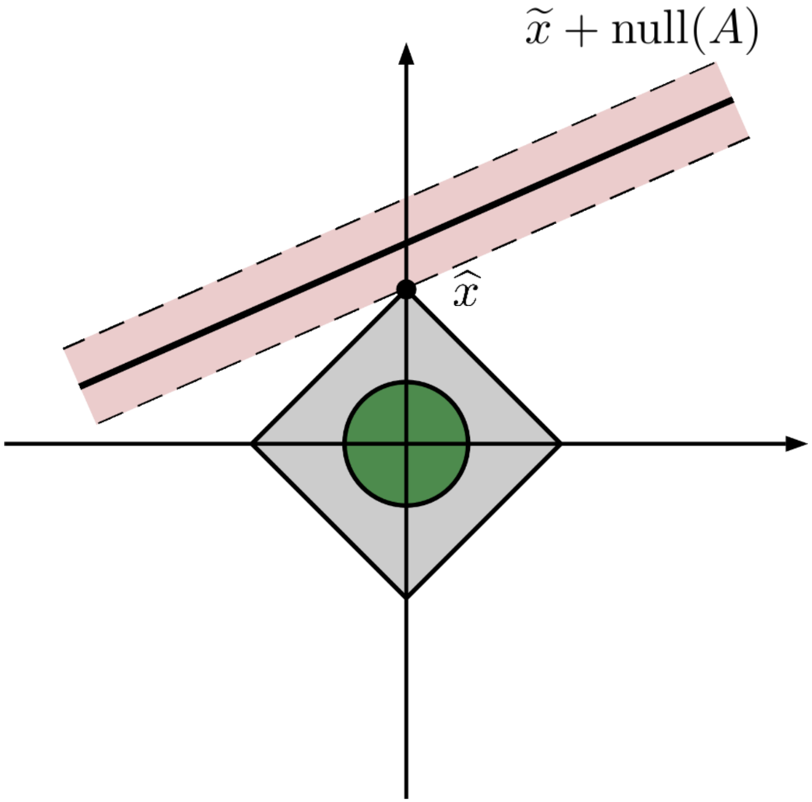

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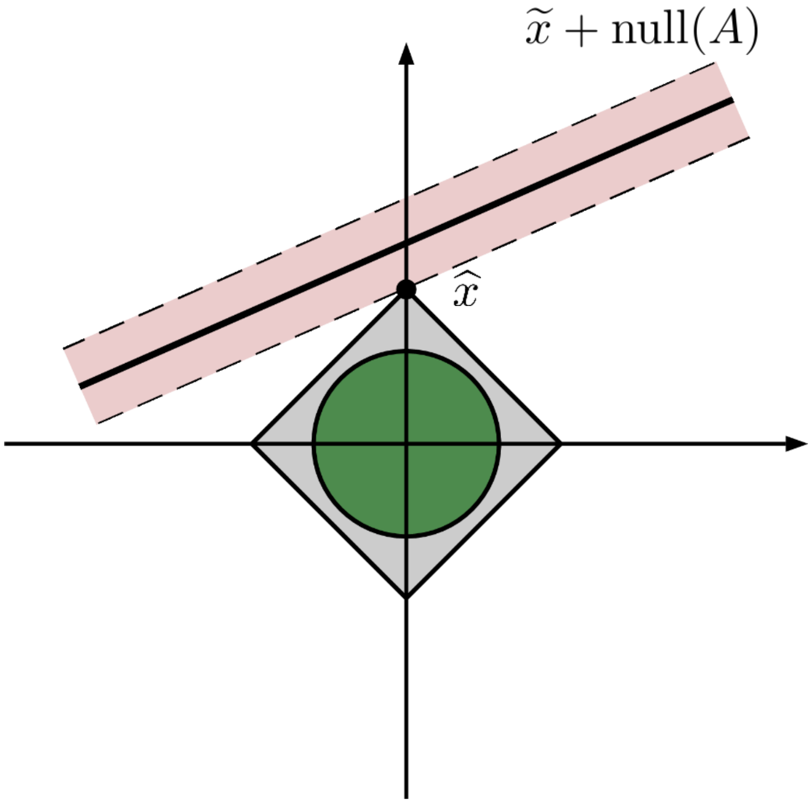

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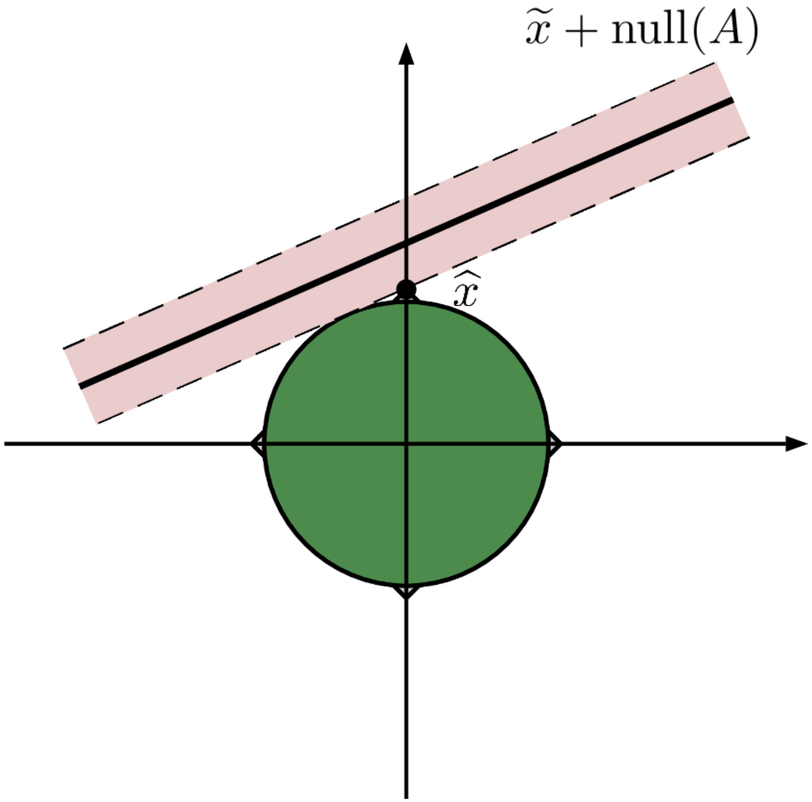


What about the L2-norm?

$$\widehat{x}_2 \in \arg\min_{x} \quad \|x\|_2^2$$
$$\text{s.t.} \quad \left\|y - Ax\right\|_2 \leq \sigma$$

L1-Norm Induces Sparsity

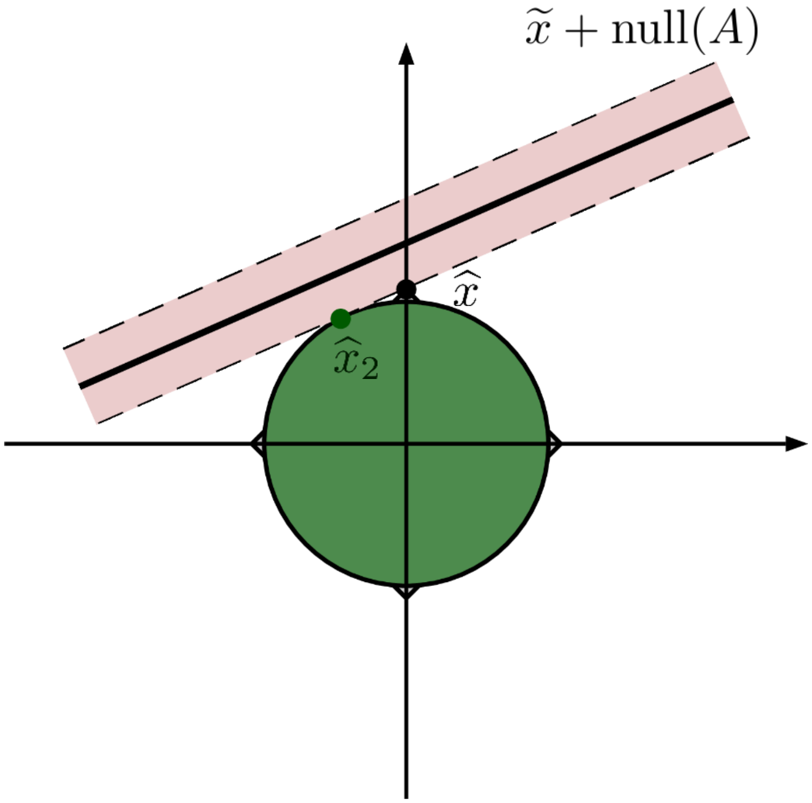
$$\widehat{x} \in \arg\min_x \|x\|_1 \quad \text{s.t.} \quad \|y - Ax\|_2 \leq \sigma$$

Assume $\sigma = 0:$

$y = Ax$ has solutions $\widetilde{x} + \mathrm{null}(A)$

$$y = A\widetilde{x} \qquad \{d : Ad = 0\}$$

Assume $\sigma > 0:$ *margin around* $\widetilde{x} + \mathrm{null}(A)$



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Example: Compressed Sensing

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sparse

$$y = \left[A : n \times d \right] x^{\star}$$

Example: Compressed Sensing

sparse

$$y = A : n \times d \quad x^\star$$

Basis Pursuit

$$\begin{aligned} \widehat{x} = \; & \arg\min_x && \|x\|_1 \\ & \text{s.t.} && y = Ax \end{aligned}$$

Example: Compressed Sensing

Compressed Sensing (CS)

sparse

$$y = A : n \times d \quad x^\star$$

Basis Pursuit

$$\widehat{x} = \arg\min_x \quad \|x\|_1 \qquad \text{s.t.} \quad y = Ax$$

Example: Compressed Sensing

Compressed Sensing (CS)

70-sparse

$$y = A : n \times d \quad x^\star$$

iid Gaussian

1000

Basis Pursuit

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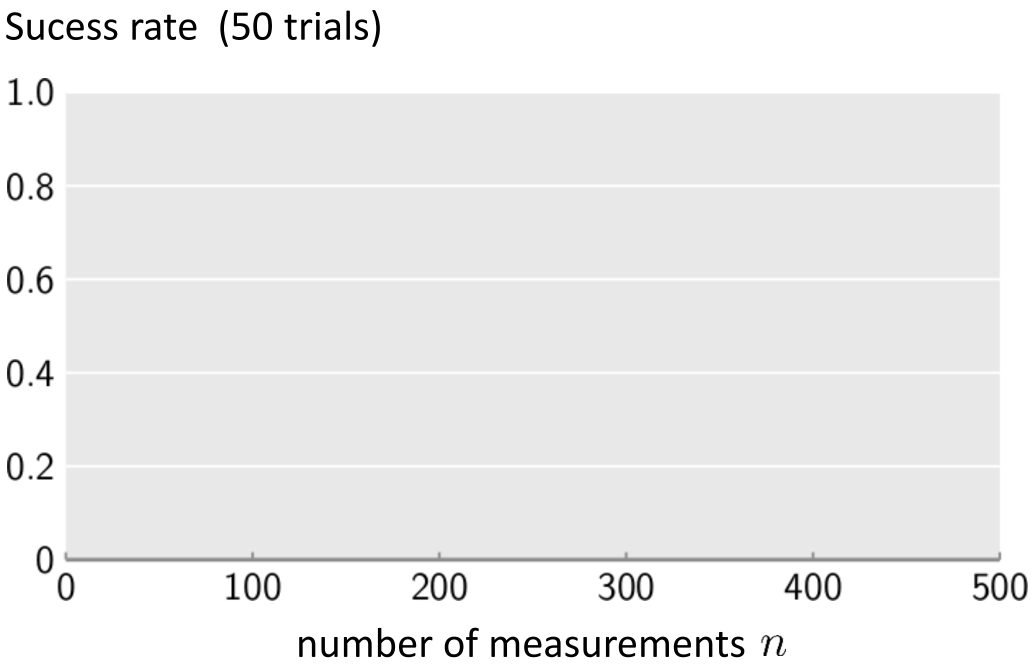
Example: Compressed Sensing



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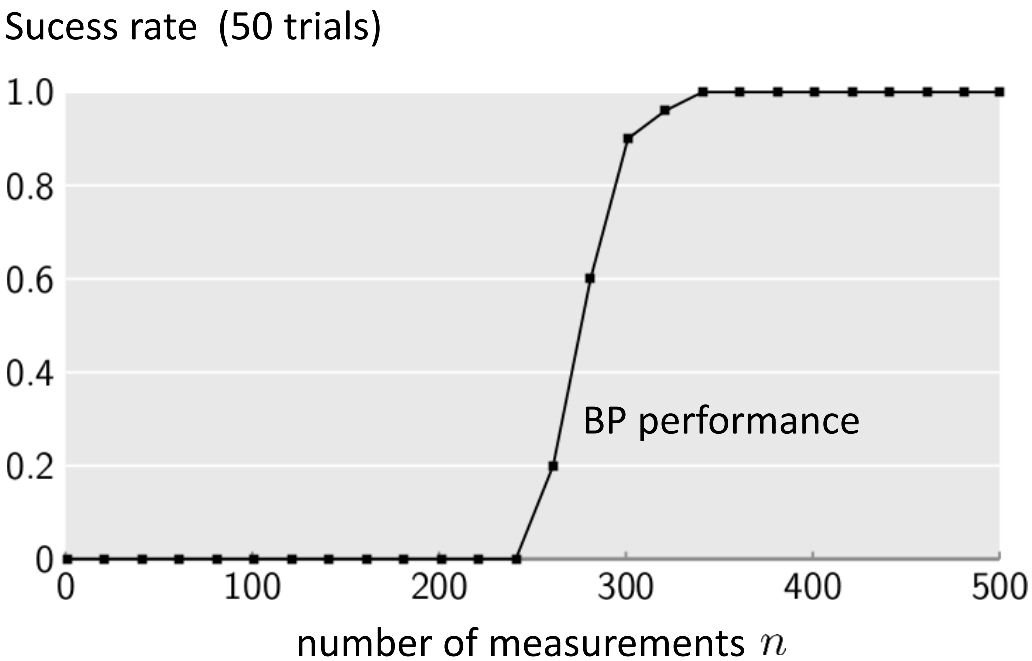

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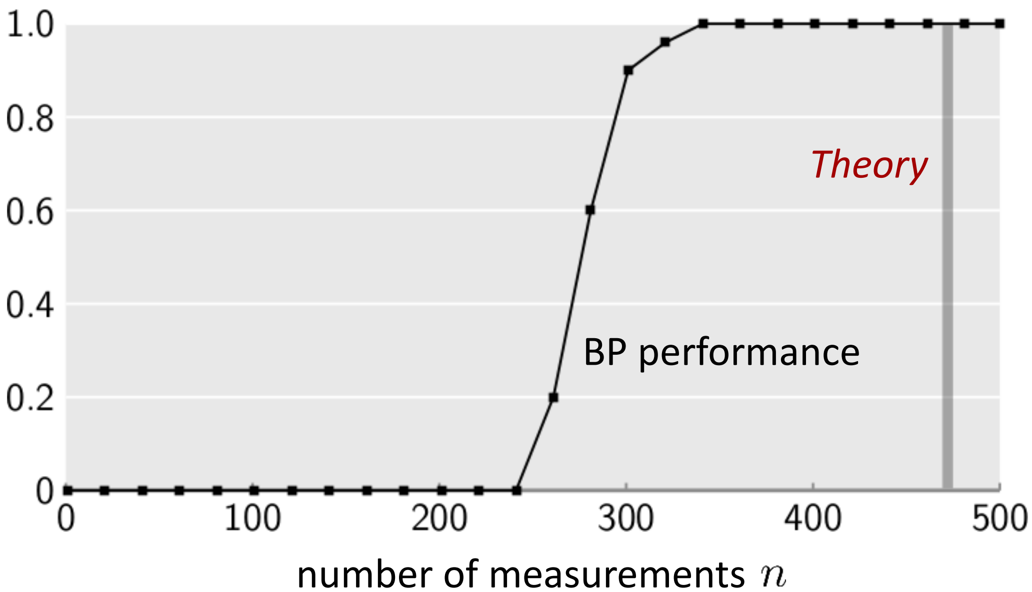
Example: Compressed Sensing

Sucess rate (50 trials)



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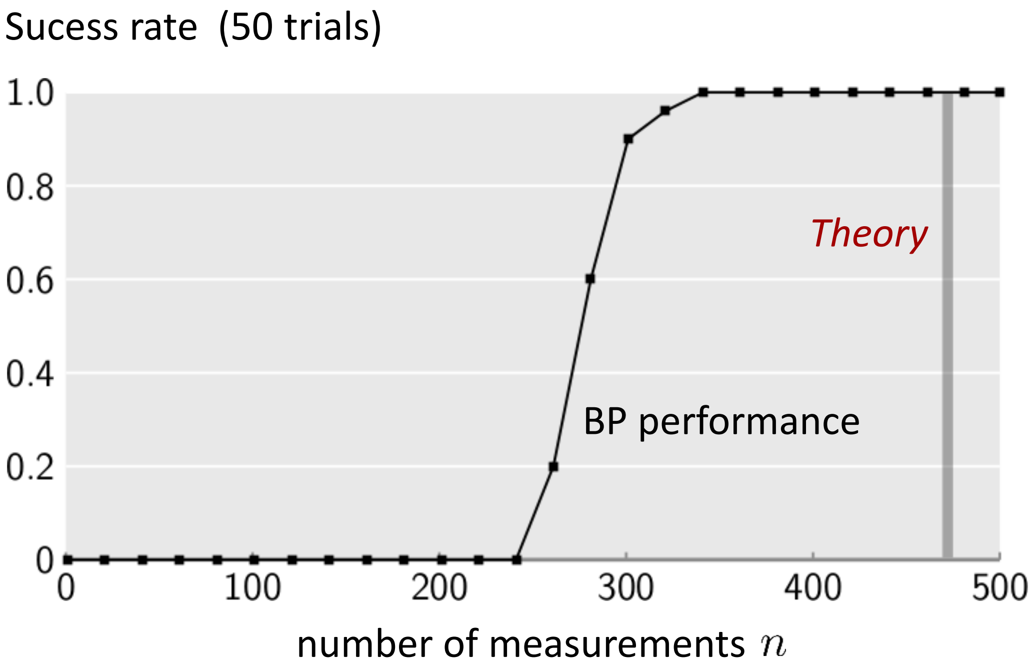

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Theorem [Chandrasekaran et al. 12’]

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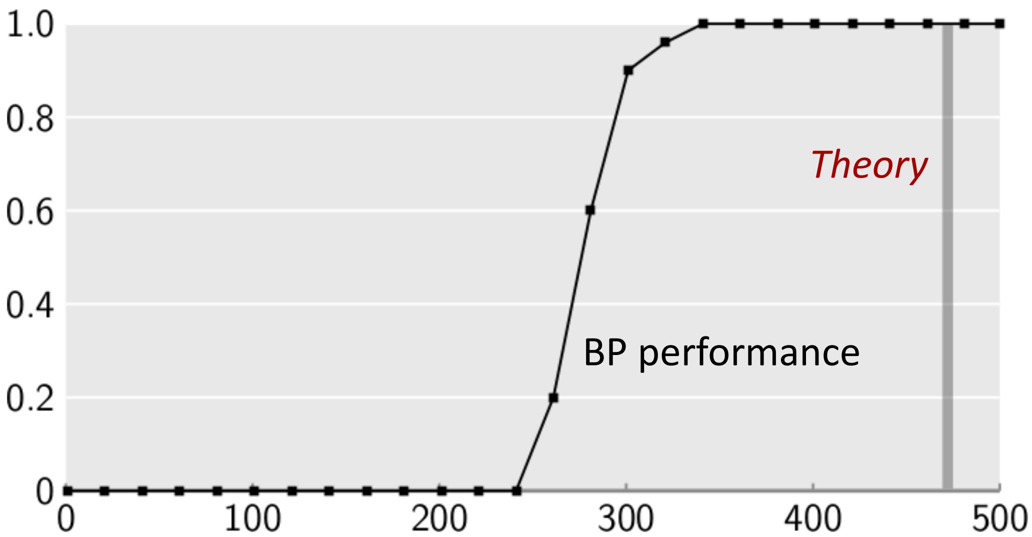

number of measurements n

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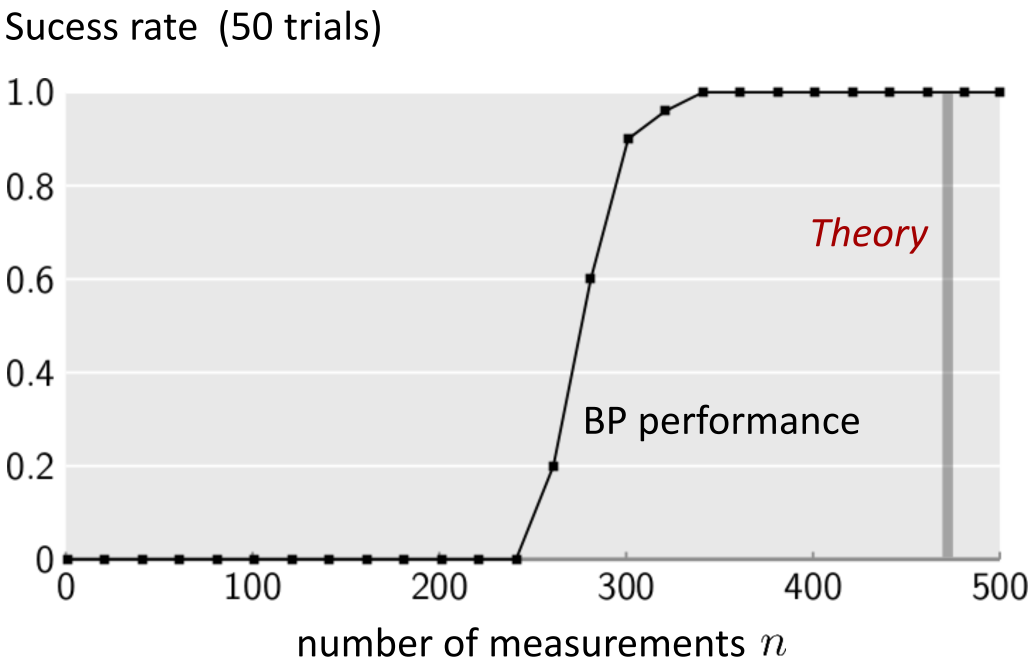

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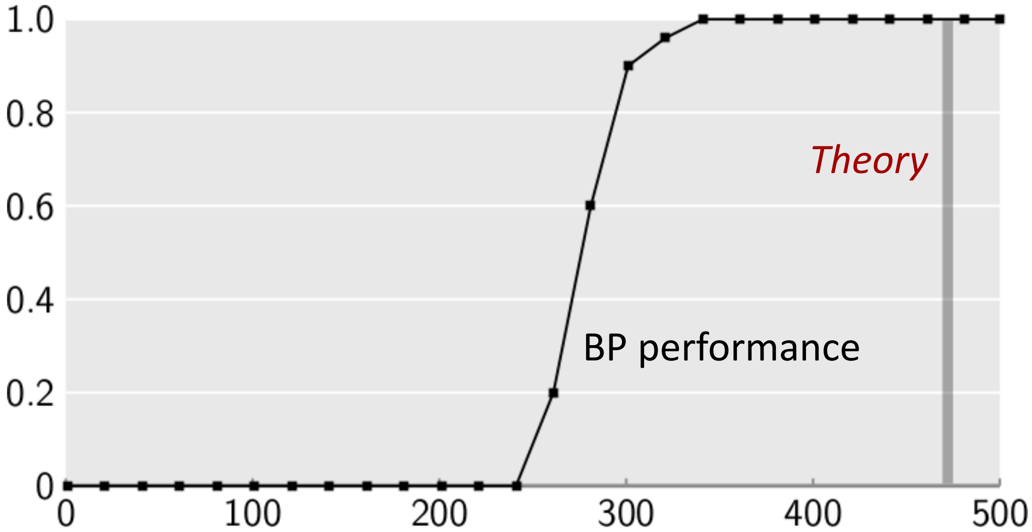

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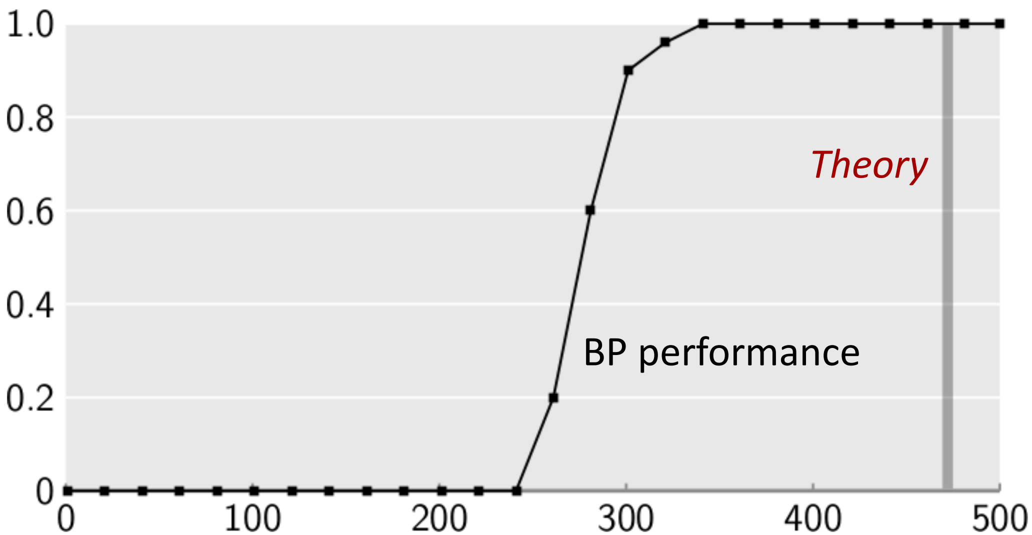
Example: Compressed Sensing

Sucess rate (50 trials)



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Theorem [Chandrasekaran et al. 12']

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iid entries $\mathcal{N}(0, 1/n)$

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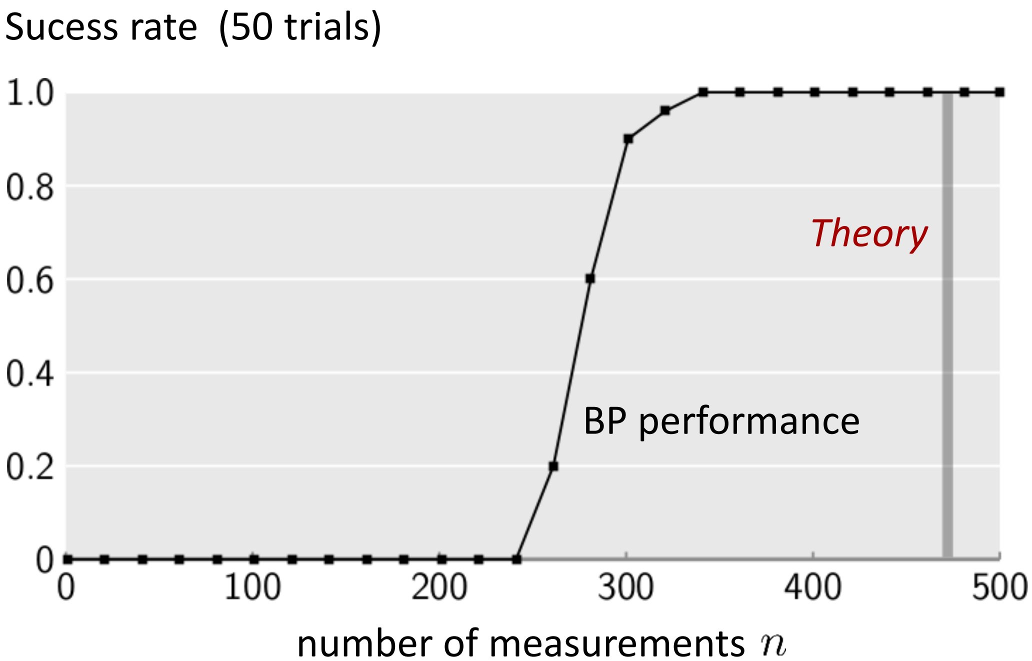

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$$n \geq 2s \log\left(\frac{d}{s}\right) + \frac{7}{5}s + 1$$

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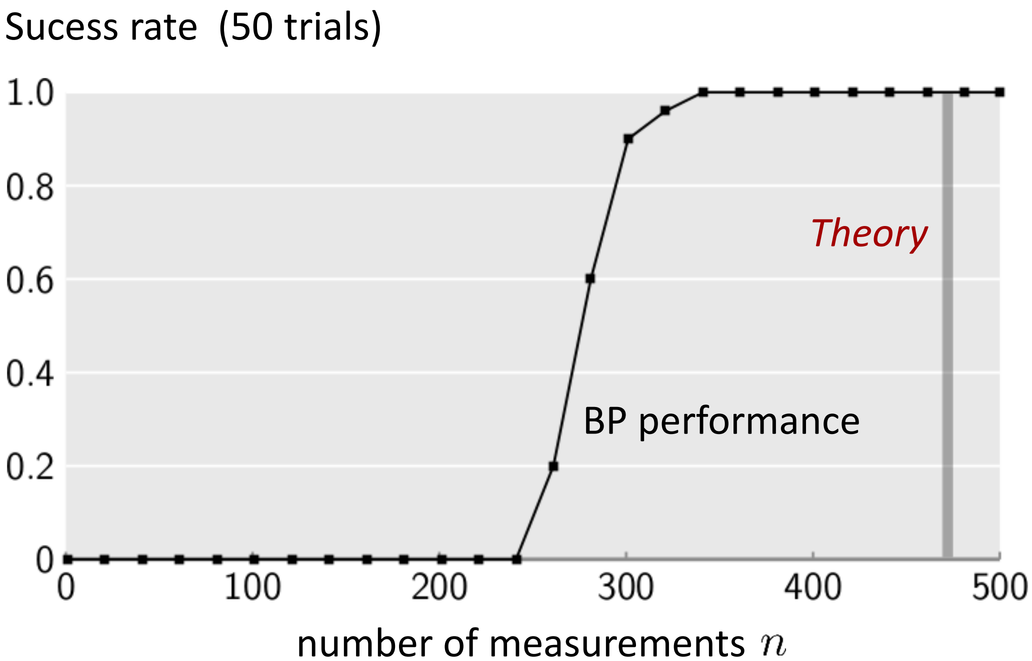

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$y = Ax^\star$ *measurements*

$$\boxed{n \geq 2s \log\left(\frac{d}{s}\right) + \frac{7}{5}s + 1} \implies x^\star = \operatorname*{argmin}_x \|x\|_1 \quad \text{s.t.} \quad y = Ax \qquad \text{w.h.p.}$$

Application: Image Reconstruction

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256×496

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$256 \times 496 \qquad \Longrightarrow \qquad z^{\star} \in \mathbb{R}^{126976}$

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Application: Image Reconstruction

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sparse or near-sparse

dictionary (wavelet, DCT, gradient space)

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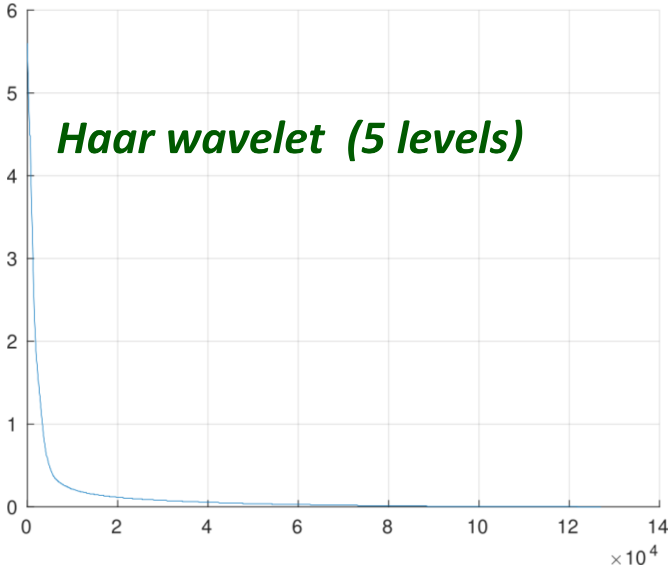
sparse or near-sparse

dictionary (wavelet, DCT, gradient space)

Ordered coefficients: $|x_i^\star|$



Application: Image Reconstruction

Suppose we observe *only 50%* of pixels

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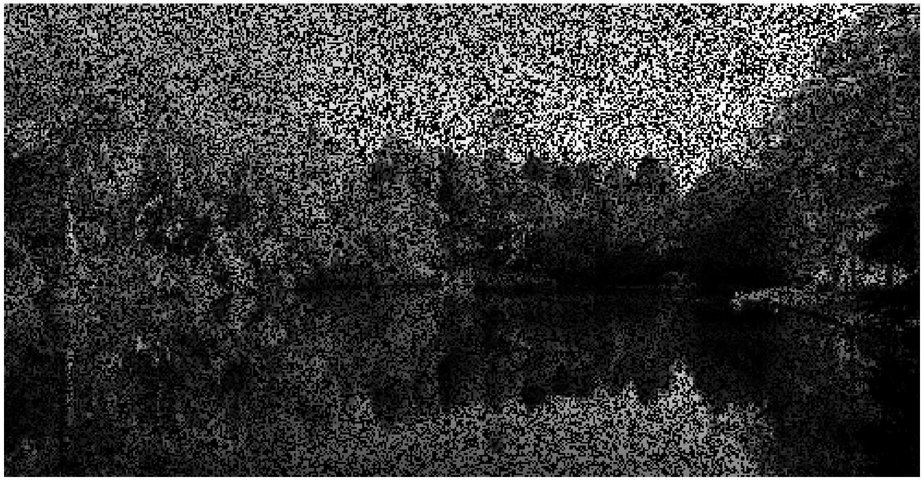
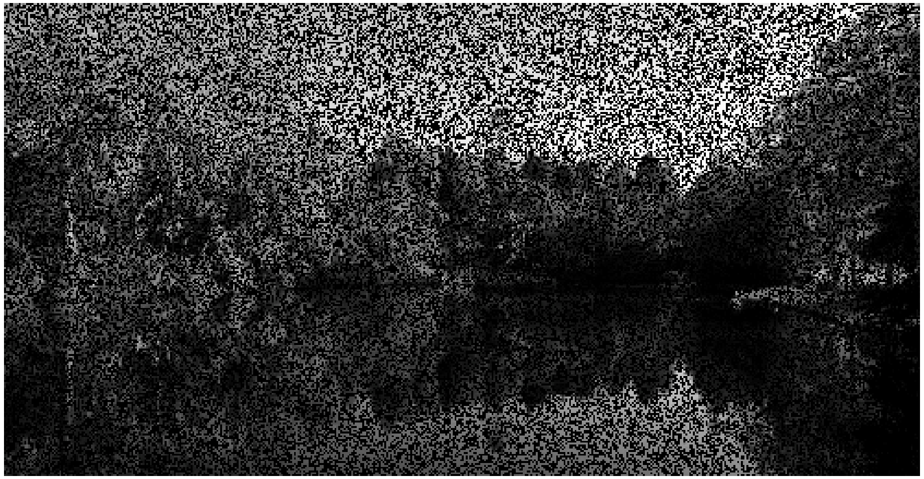


y

Application: Image Reconstruction

Suppose we observe *only 50%* of pixels



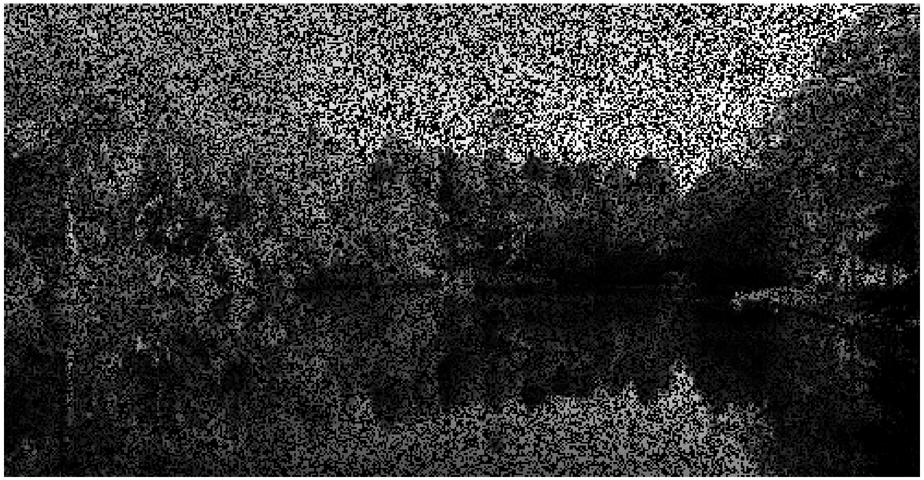
y

Solve

$$\widehat{x} = \arg\min_{x} \; \|x\|_1 \quad \text{s.t.} \quad y = \Phi \, \Psi \, x$$

Application: Image Reconstruction

Suppose we observe *only 50%* of pixels



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wavelet

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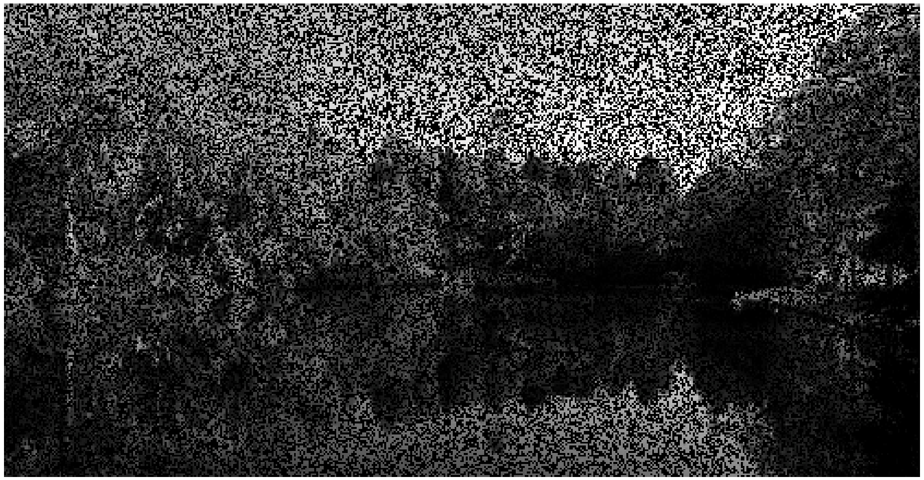

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wavelet

observed indices

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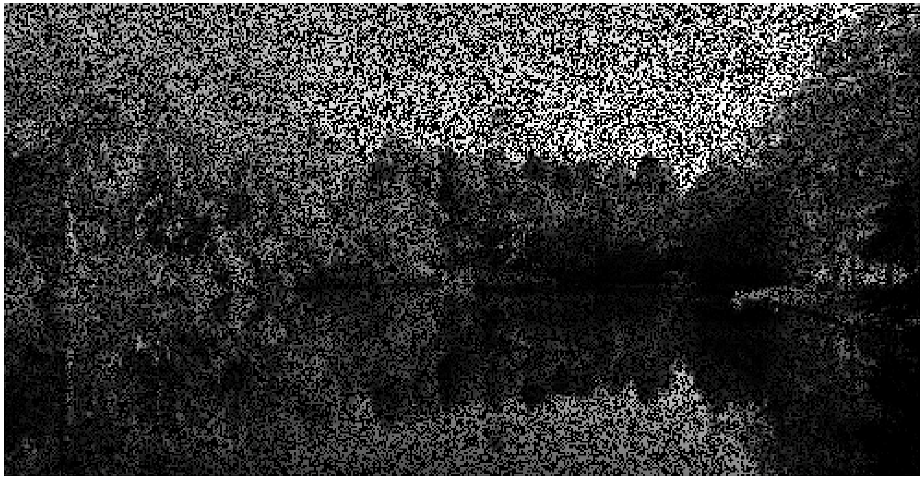

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wavelet

observed indices

$\widehat{x}$

Application: Image Reconstruction

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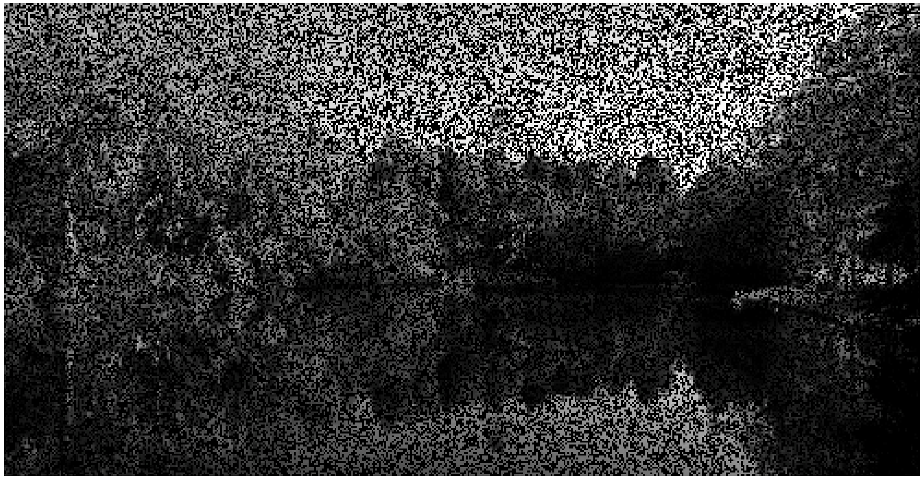


y

Solve

$$\widehat{x} = \arg\min_{x} \|x\|_1 \quad \text{s.t.} \quad y = \underbrace{\Phi}_{\text{observed indices}} \underbrace{\Psi}_{\text{wavelet}} x$$

$\widehat{x}$

PSNR: 21.31 dB

Application: Image Reconstruction

Solve

$$\widehat{x} = \underset{x}{\arg\min} \ \|x\|_1 \quad \text{s.t.} \quad y = \underbrace{\Phi}_{\textit{partial DFT}} \underbrace{\Psi}_{\text{wavelet}} x$$

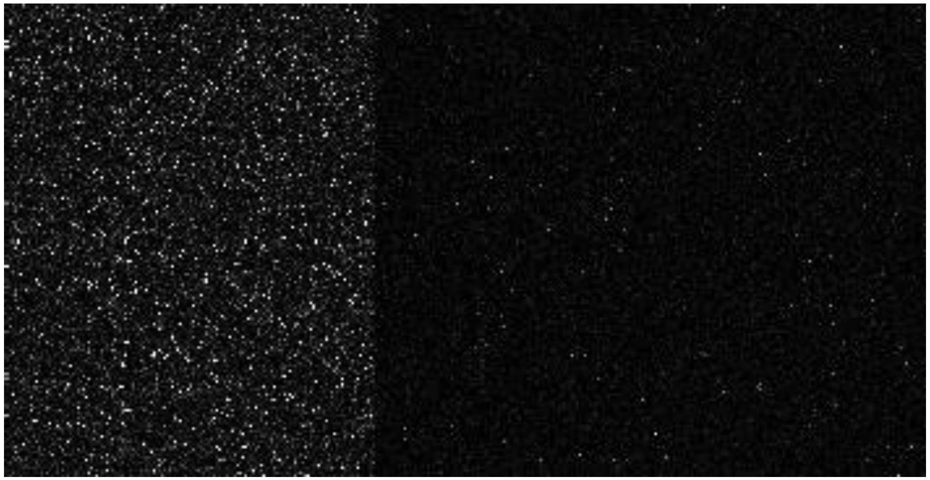
Application: Image Reconstruction

Solve

$$\widehat{x} = \arg\min_{x} \|x\|_1 \quad \text{s.t.} \quad y = \underbrace{\Phi}_{\textit{partial DFT}} \underbrace{\Psi}_{\text{wavelet}} x$$

each entry of y has info from entire image

Application: Image Reconstruction



y

Solve

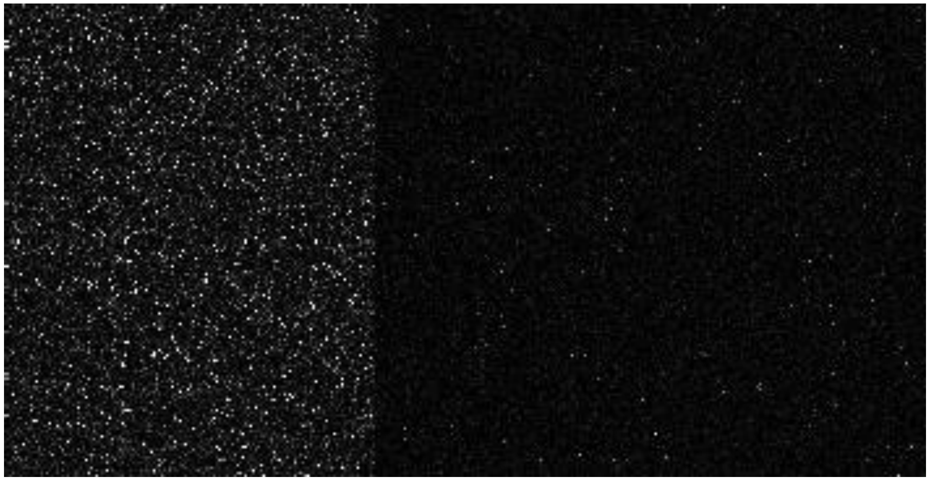
$$\widehat{x} = \arg\min_{x} \quad \|x\|_1 \quad \text{s.t.} \quad y = \Phi\,\Psi\,x$$

wavelet

partial DFT

each entry of y *has info from entire image*

Application: Image Reconstruction



y

Solve

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wavelet

partial DFT

each entry of y has info from entire image

$\widehat{x}$

PSNR: 24.93 dB

Outline

- *Motivation: Hypothesis Testing in High-Dimensions*
- *Introduction to LASSO and other sparsity problems*
- Gaussian graphical model selection
- Matrix completion

Gaussian Graphical Model Selection

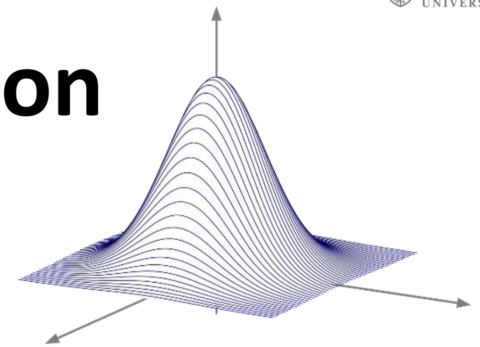
Gaussian Graphical Model Selection

$X = (X_1, X_2, \ldots, X_d)$

Gaussian Graphical Model Selection

$$X = (X_1, X_2, \ldots, X_d) \;\sim\; \mathcal{N}\big(0_d\,,\,\Sigma^\star\big)$$

Gaussian Graphical Model Selection



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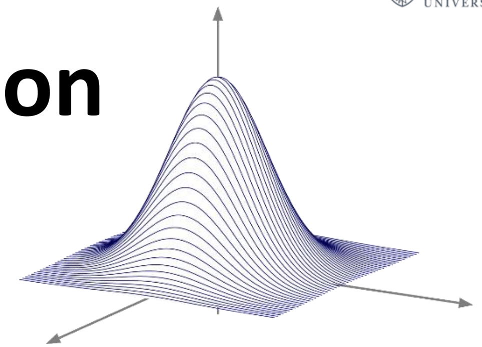
Gaussian Graphical Model Selection



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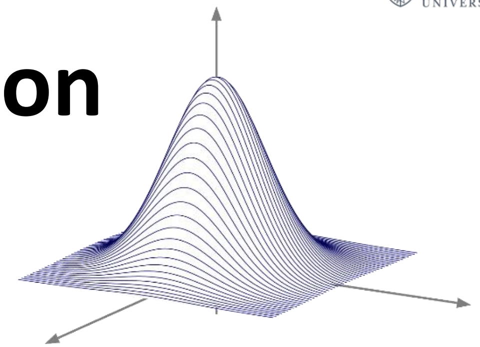
<u>Problem:</u> given n idd observations of X, denoted $X^{(1)}, \ldots, X^{(n)}$, estimate $\Sigma^\star$

Gaussian Graphical Model Selection



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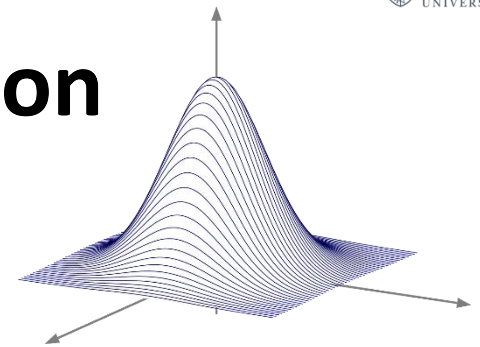

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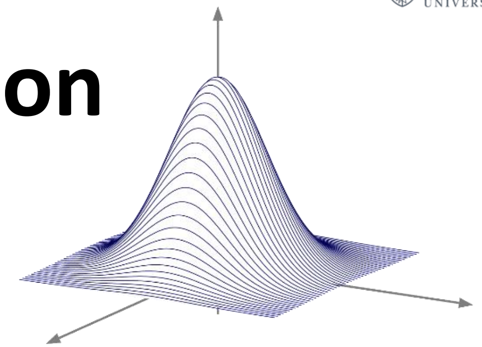
Assumption: most pairs of coordinates (X_i, X_j) are conditionally independent

Gaussian Graphical Model Selection



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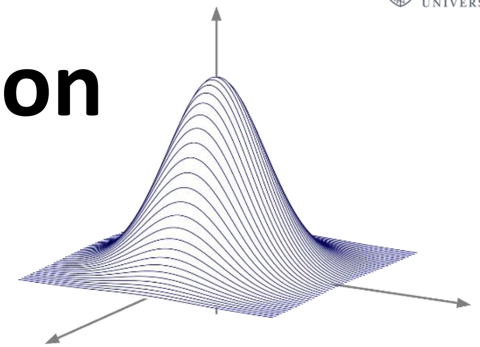
$\Downarrow$

precision matrix $\Theta^\star := \big(\Sigma^\star\big)^{-1}$ is *sparse*

Gaussian Graphical Model Selection



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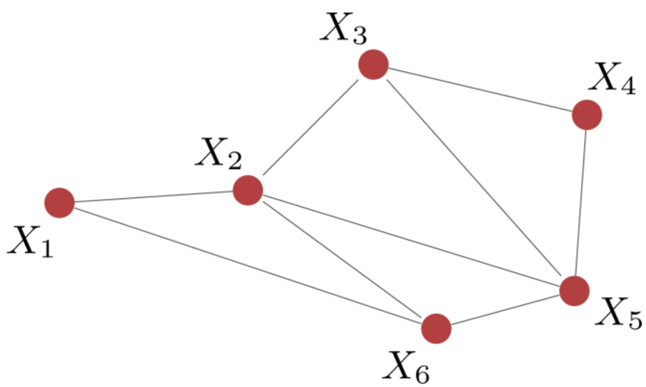

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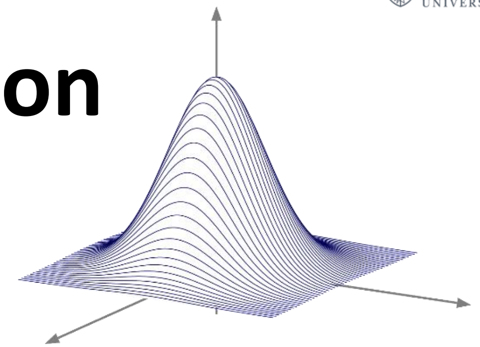

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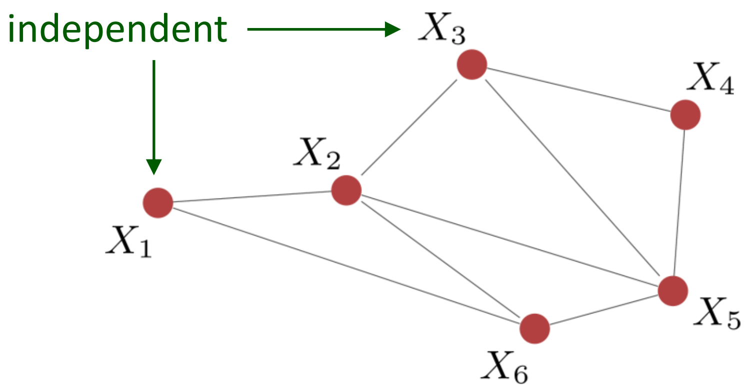
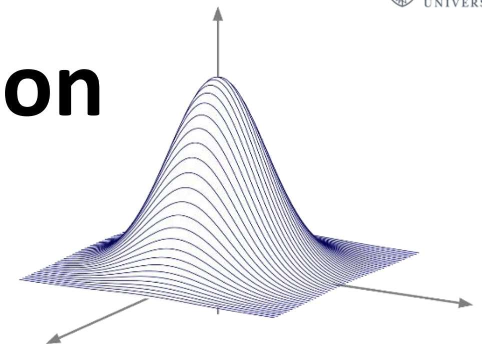


Gaussian Graphical Model Selection



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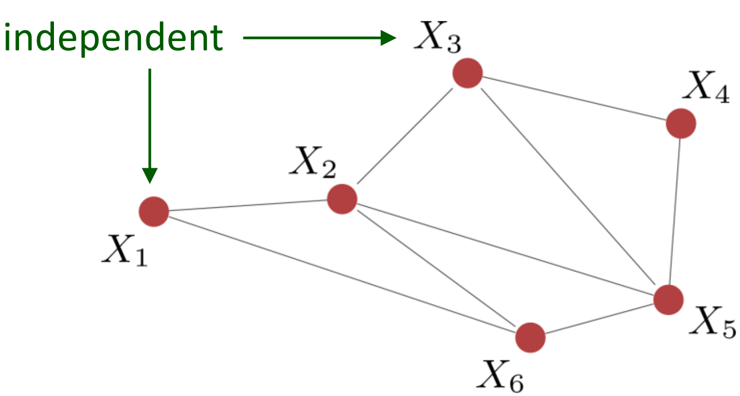
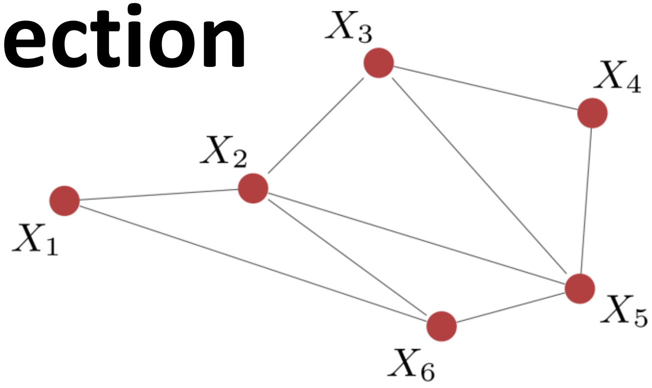
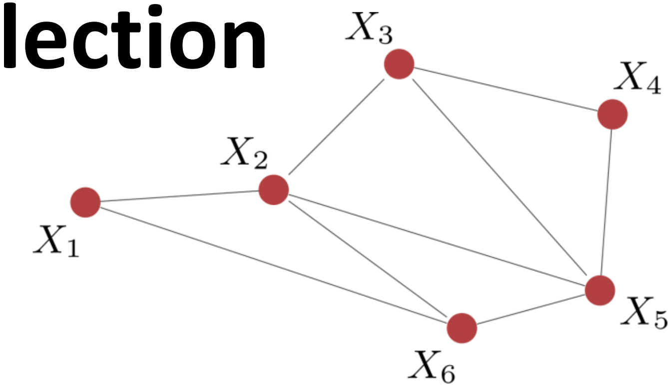


$$\Longrightarrow \qquad \Theta^\star = \begin{bmatrix} \star & \star & 0 & 0 & 0 & \star \\ \star & \star & \star & 0 & \star & \star \\ 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & 0 \\ 0 & \star & \star & \star & \star & \star \\ \star & \star & 0 & 0 & \star & \star \end{bmatrix}$$

Gaussian Graphical Model Selection



Gaussian Graphical Model Selection



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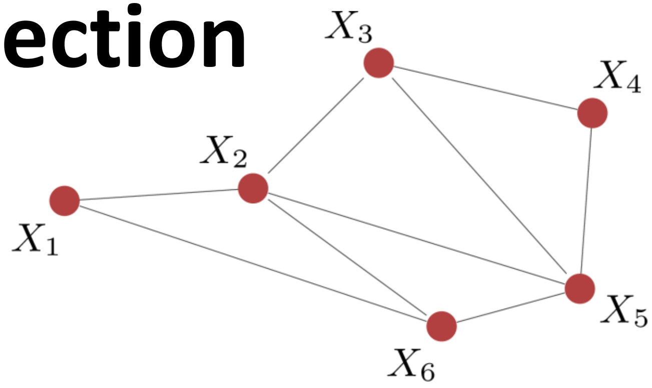

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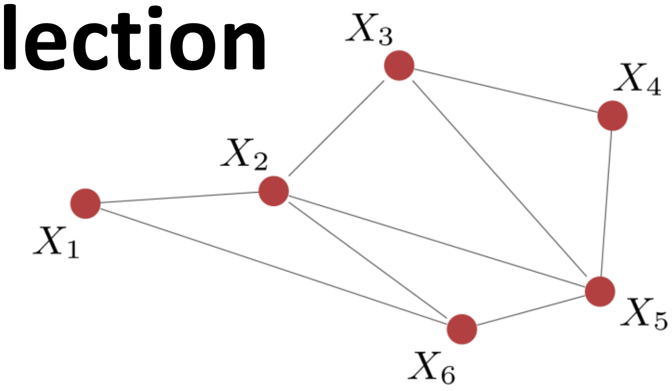
pdf : $$f_X\big(x\,;\,\Theta^\star\big) = \frac{\sqrt{\det\ \Theta^\star}}{\big(2\pi\big)^{\frac{d}{2}}} \exp\Big(-\frac{1}{2}x^\top\Theta^\star x\Big)$$

Gaussian Graphical Model Selection



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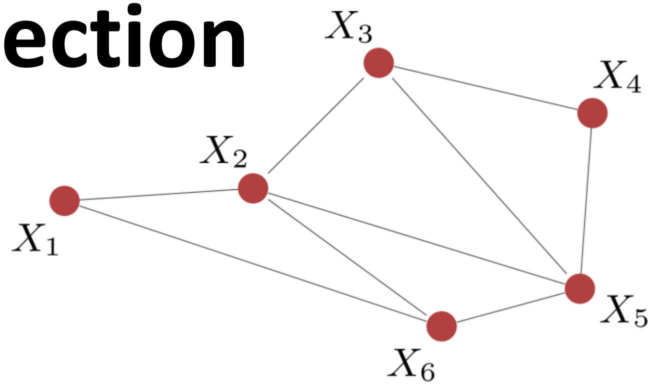

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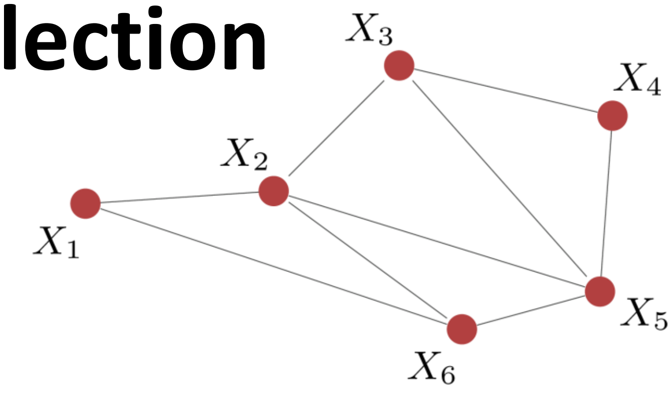

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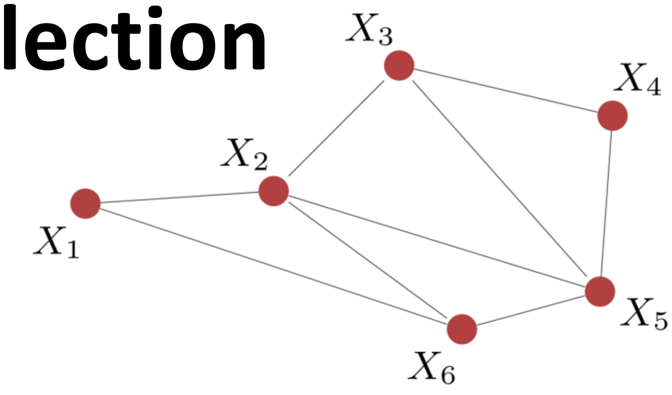

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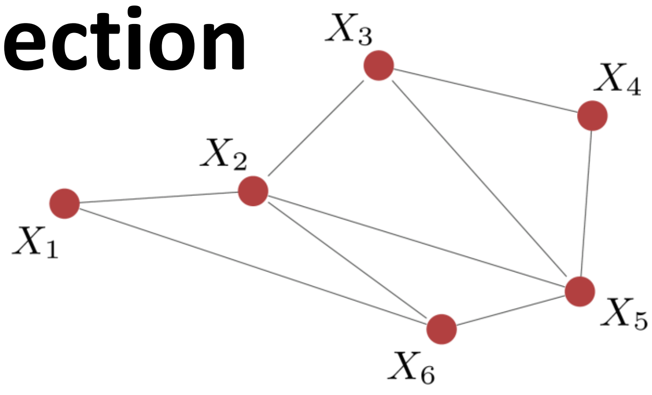
sample covariance matrix

Gaussian Graphical Model Selection



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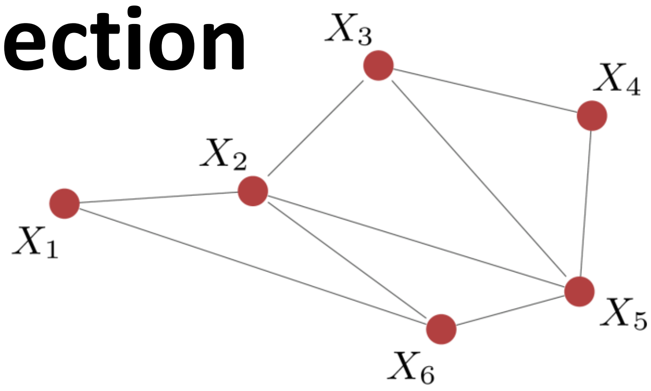
Gaussian Graphical Model Selection



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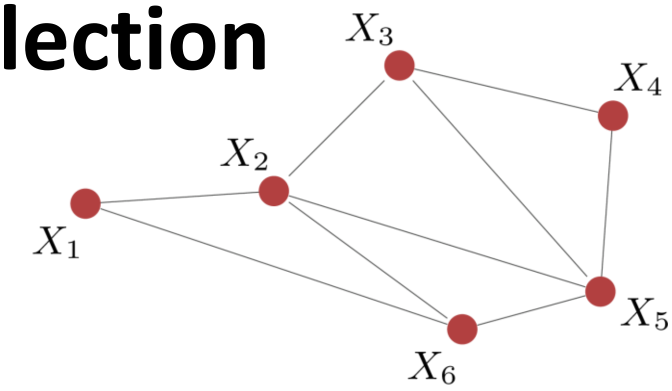
Gaussian Graphical Model Selection



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(assuming it is invertible $\implies$ $n > d$ *)*

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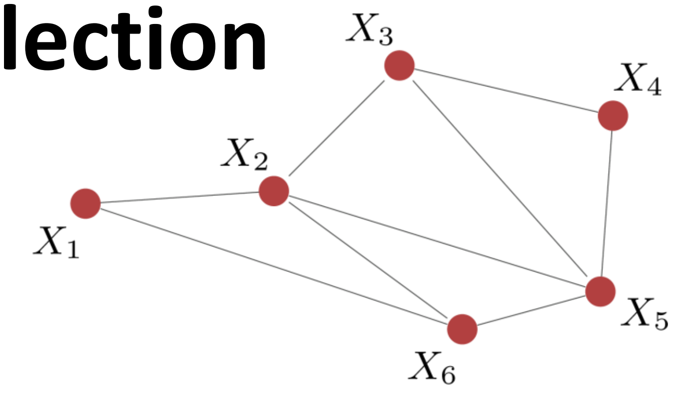

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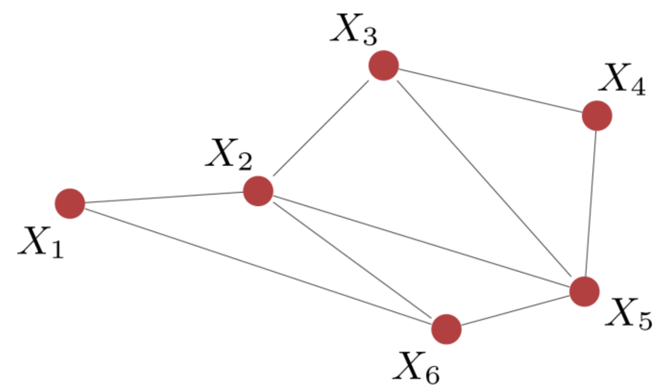
Graphical LASSO



Maximum likelihood estimator

$$\widehat{\Theta}^{\star}_{\mathrm{ML}} = \arg\min_{\Theta} - \log\,\det\,\Theta + \mathrm{tr}\Big(\Theta\,\widehat{\Sigma}_n\Big)$$

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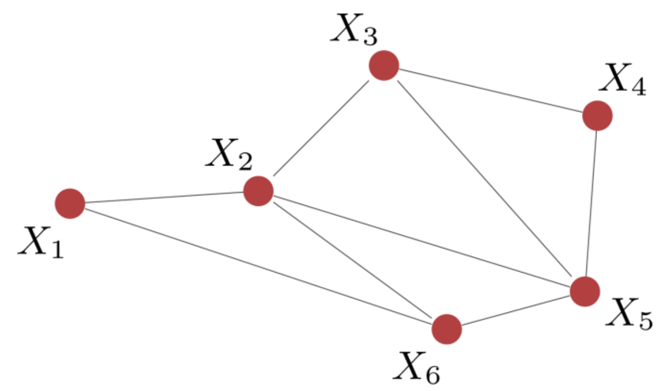

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Assumption: most pairs of coordinates (X_i, X_j) are conditionally independent

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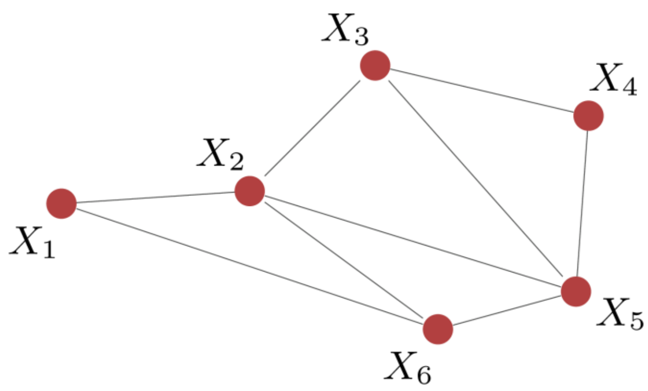

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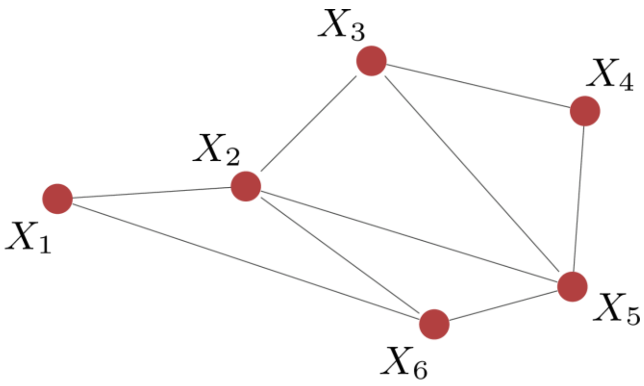

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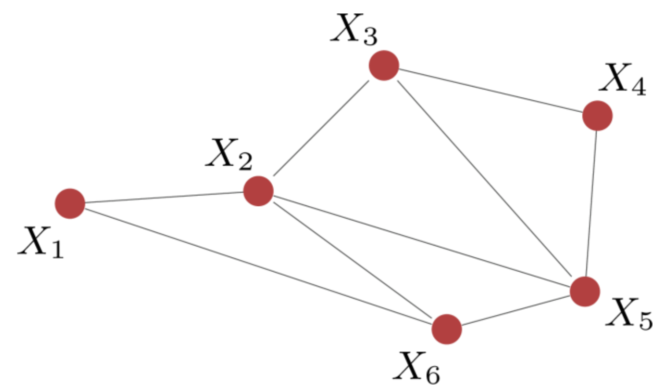
Graphical LASSO



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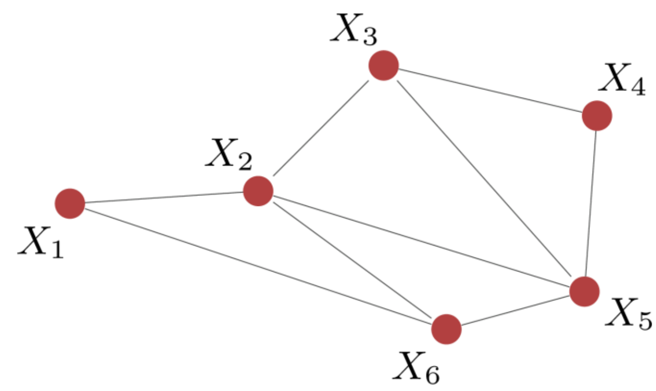

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applies L1-norm only to off-diagonal entries

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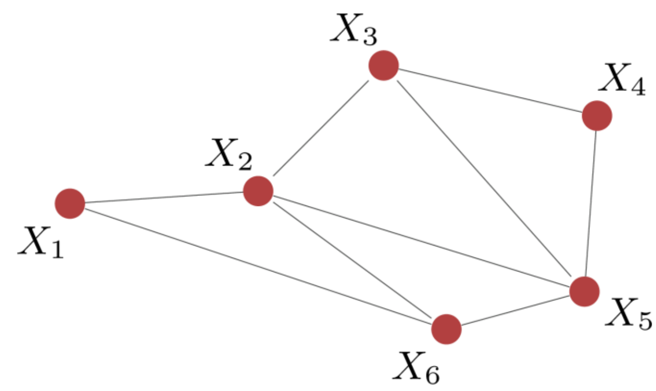

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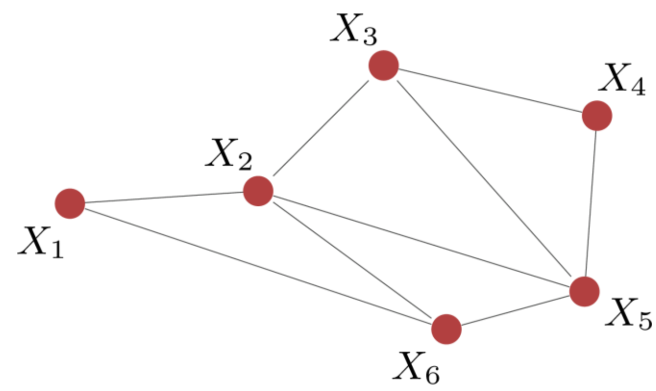
$$\Theta^{\star} = \begin{bmatrix} \star & \star & 0 & 0 & 0 & \star \\ \star & \star & \star & 0 & \star & \star \\ 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & 0 \\ 0 & \star & \star & \star & \star & \star \\ \star & \star & 0 & 0 & \star & \star \end{bmatrix}$$

Graphical LASSO



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sensible estimators even for non-Gaussian RVs

Example: Dow Jones

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Price of stock of *6 companies* at beginning of each week of 2011 (Jan-Jun)

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Week	Alcoa	American Express	Boeing	Bank of America	Caterpillar	Cisco Systems
1	14.67	43.30	66.15	10.59	100.25	14.94
2	15.29	43.73	69.26	10.89	101.30	15.14
3	15.82	43.86	69.42	11.18	102.59	16.04
4	15.87	43.86	70.29	11.47	102.72	16.41
5	15.92	43.96	70.86	11.87	103.42	16.59
6	15.95	44.13	71.17	11.89	103.56	16.82
7	15.96	44.20	71.43	12.28	104.86	16.88
8	16.18	44.75	71.52	12.32	105.58	16.93
9	16.19	44.94	71.60	12.36	105.87	17.01
⋮	⋮	⋮	⋮	⋮	⋮	⋮
24	17.42	50.74	79.31	14.77	96.93	21.22
25	18.06	51.39	80.35	15.08	99.62	22.11

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remove mean	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$

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$$\widetilde{X}_1 \quad \widetilde{X}_2 \quad \widetilde{X}_3 \quad \widetilde{X}_4 \quad \widetilde{X}_5 \quad \widetilde{X}_6$$

remove mean

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$X_1 \quad X_2 \quad X_3 \quad X_4 \quad X_5 \quad X_6$$

$d = 6$

$n = 25$

Example: Dow Jones

Example: Dow Jones

$$\widehat{\Theta}_{\mathrm{ML}} = \widehat{\Sigma}_n^{-1} = \begin{bmatrix} 60.82 & 4.85 & -6.21 & -21.34 & -0.07 & -4.81 \\ 4.85 & 7.34 & -1.22 & -5.50 & 0.37 & -4.78 \\ -6.21 & -1.22 & 3.03 & 2.08 & -0.08 & -2.69 \\ -21.34 & -5.50 & 2.08 & 14.31 & -0.42 & 1.69 \\ -0.07 & 0.37 & -0.08 & -0.42 & 0.06 & 0.07 \\ -4.81 & -4.78 & -2.69 & 1.69 & 0.07 & 11.38 \end{bmatrix}$$

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Boeing

Bank of America

American Express

Caterpillar

Alcoa

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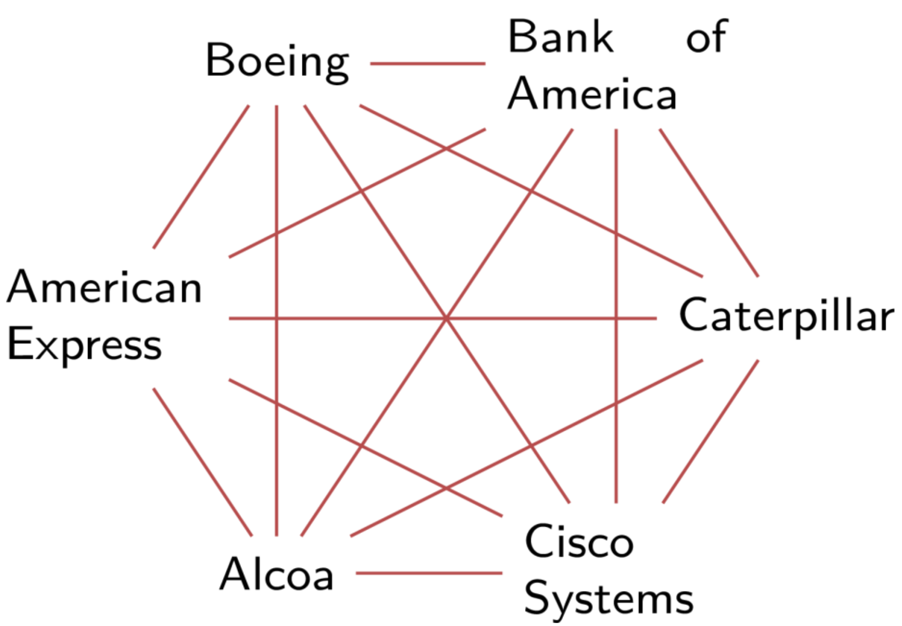

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graphical LASSO

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graphical LASSO

$\left|(\widehat{\Theta}_{\mathrm{GL}})_{ij}\right| \leq 10^{-3} \quad \Longrightarrow$ we assume no correlation, i.e., *no edge* (i, j)

Example: Dow Jones

$$\widehat{\Theta}_{\mathrm{GL}} = \arg\min_{\Theta} - \log\det\Theta + \mathrm{tr}\Big(\Theta\,\widehat{\Sigma}_n\Big) + \lambda\big\|\Theta\big\|_{1,\text{off-d}}$$

graphical LASSO

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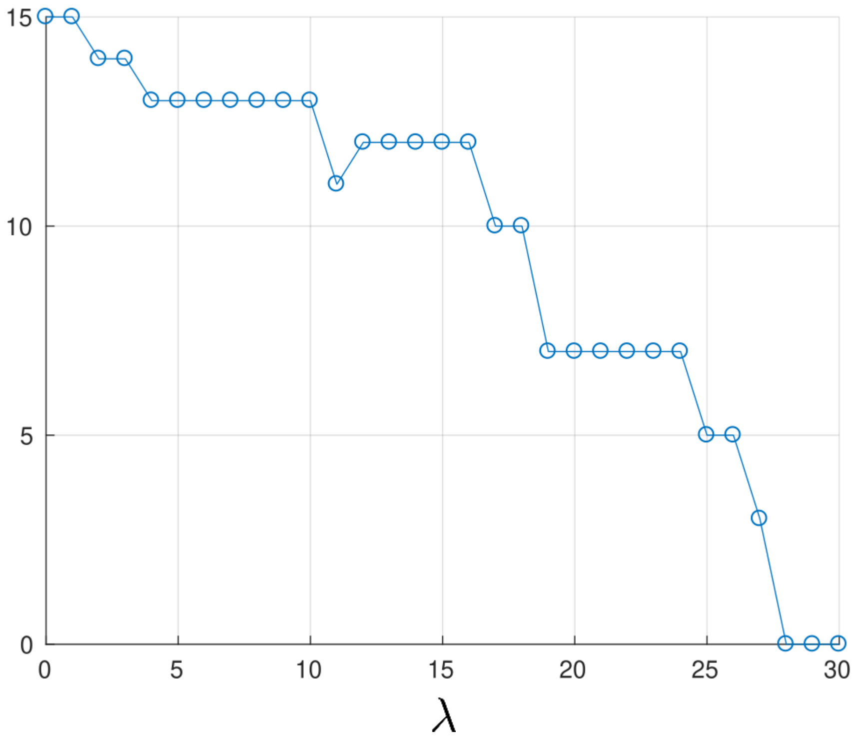
Number of edges



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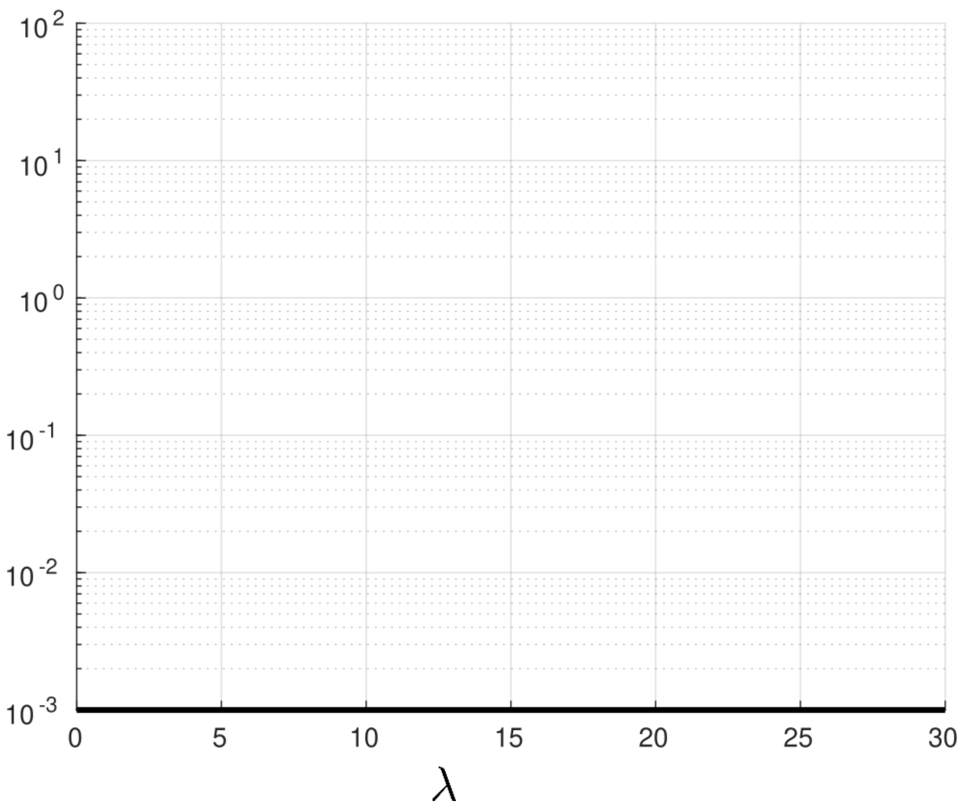

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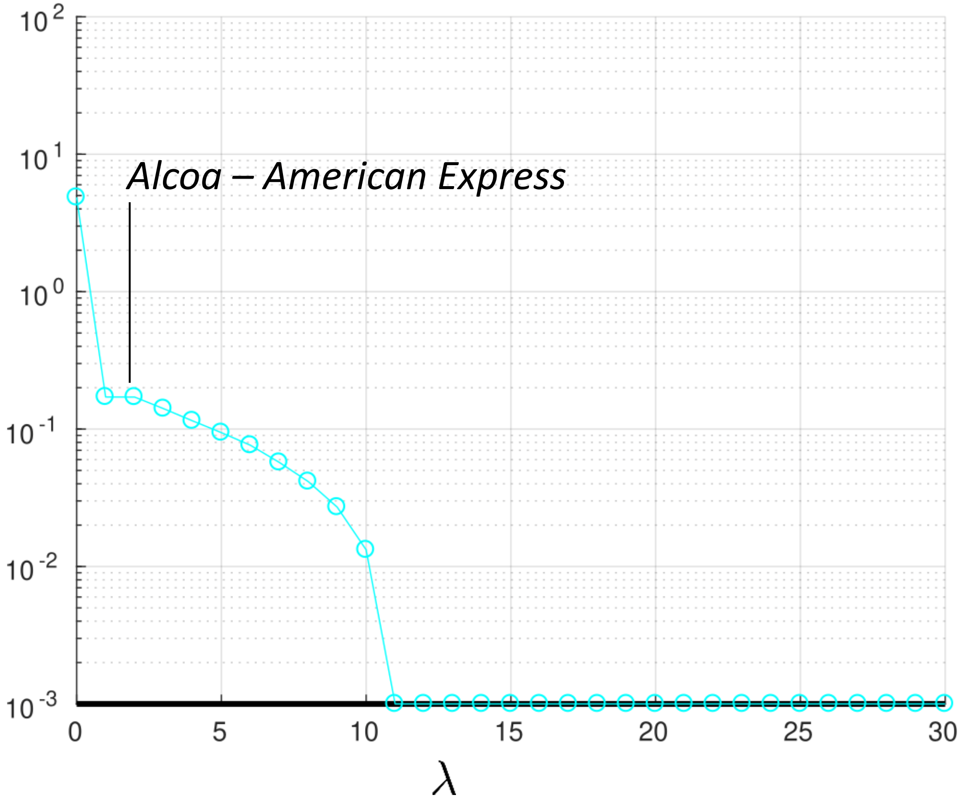

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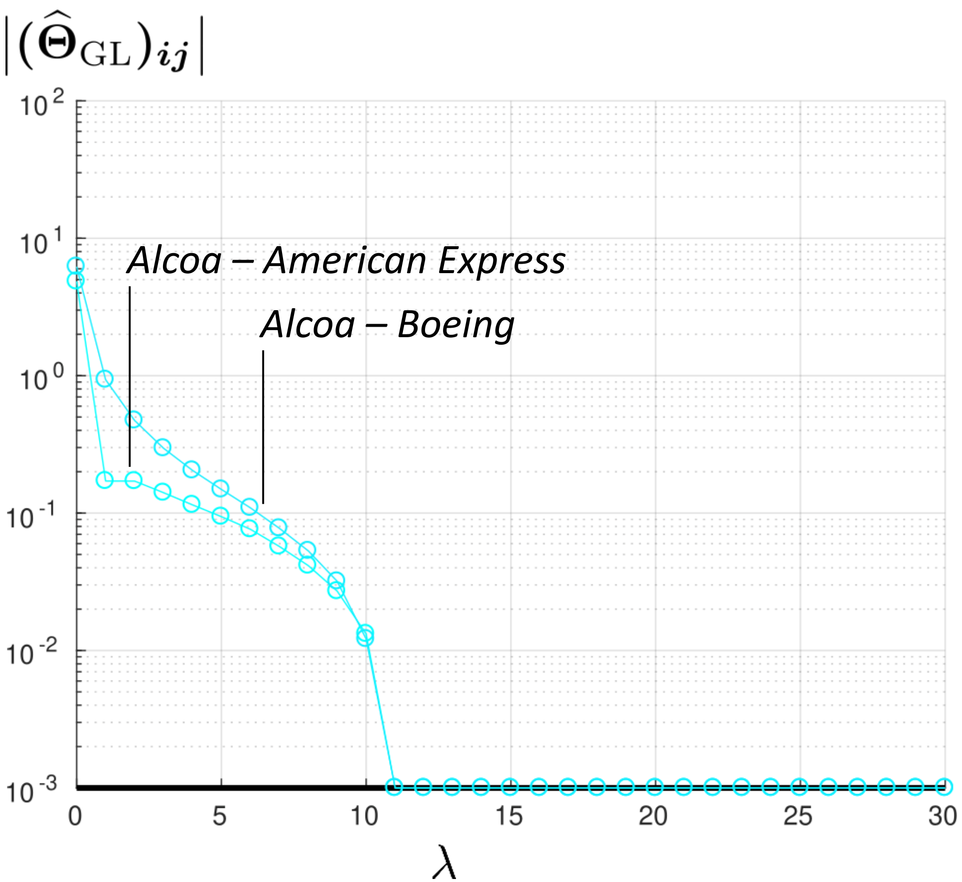

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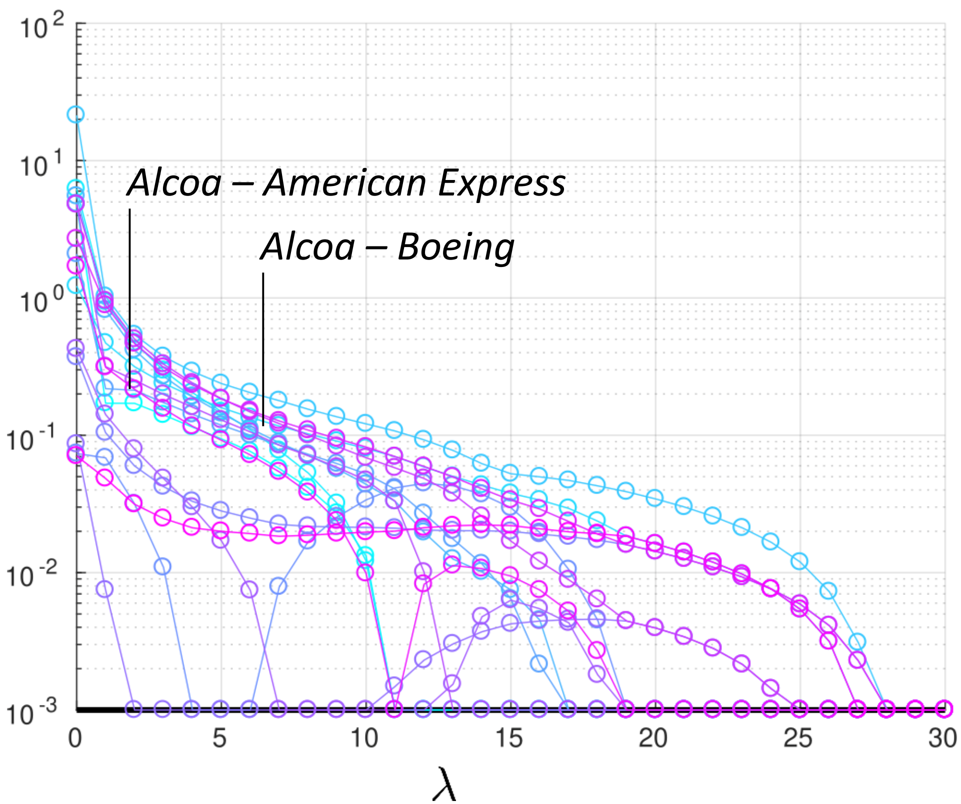

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$\left|(\widehat{\Theta}_{\mathrm{GL}})_{ij}\right| \leq 10^{-3} \implies$ we assume no correlation, i.e., *no edge* (i,j)

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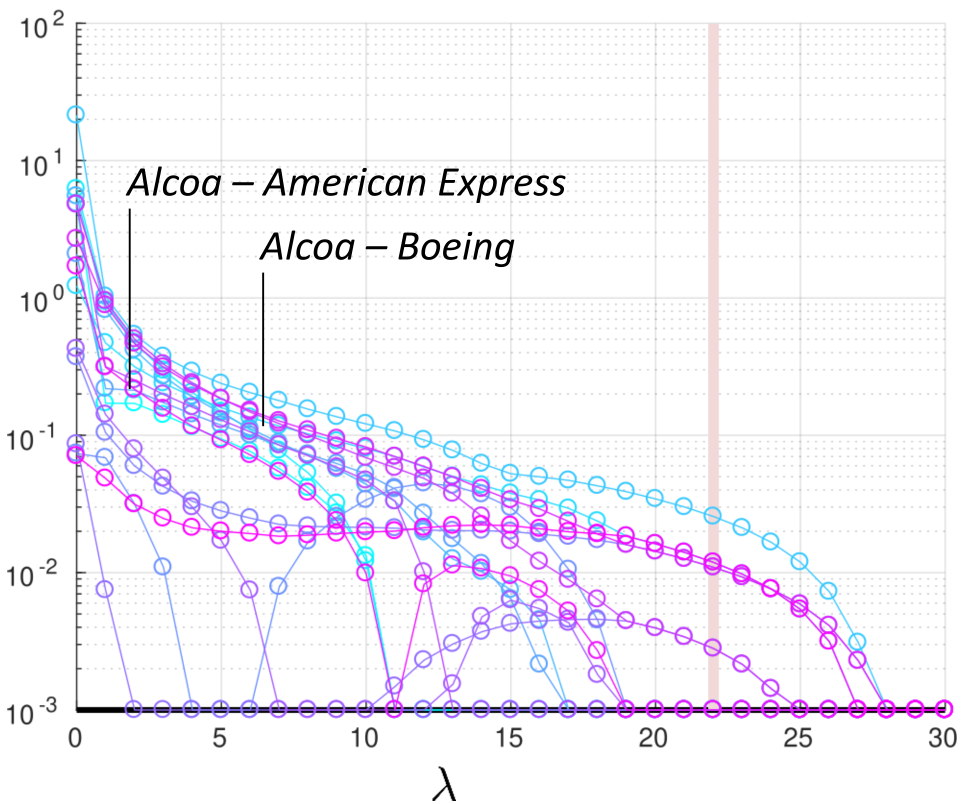

Example: Dow Jones

$$\widehat{\Theta}_{\mathrm{GL}} = \begin{bmatrix} 1.814 & 0 & 0 & -0.030 & 0 & 0 \\ 0 & 0.186 & -0.014 & 0 & 0.013 & -0.003 \\ 0 & -0.014 & 0.096 & 0 & 0.003 & -0.013 \\ -0.030 & 0 & 0 & 0.601 & 0 & 0 \\ 0 & 0.013 & 0.003 & 0 & 0.026 & 0.014 \\ 0 & -0.003 & -0.013 & 0 & 0.014 & 0.282 \end{bmatrix}$$

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Boeing

Bank of America

American Express

Caterpillar

Alcoa

Cisco Systems

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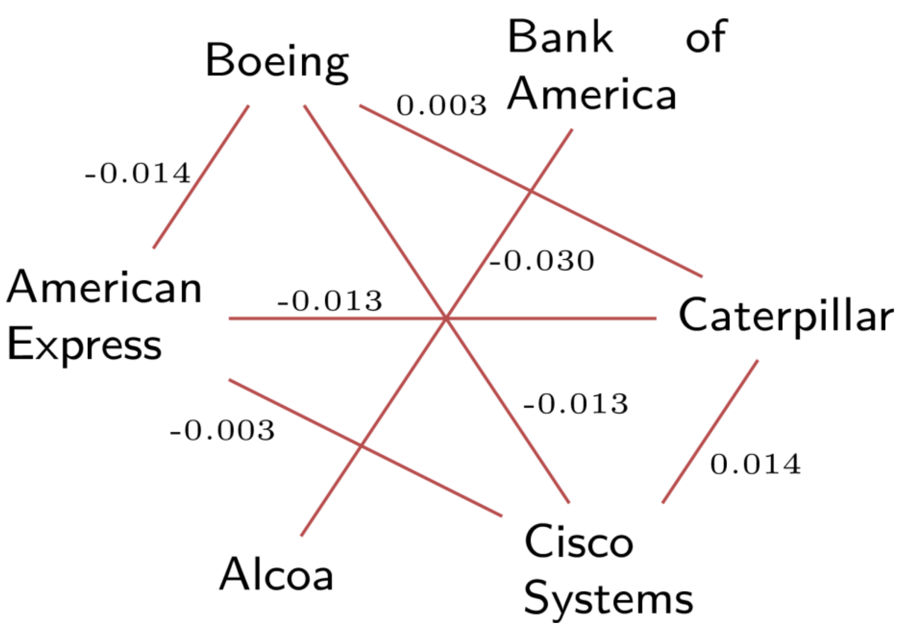

Outline

- *Motivation: Hypothesis Testing in High-Dimensions*
- *Introduction to LASSO and other sparsity problems*
- *Gaussian graphical model selection*
- Matrix completion

Matrix Completion

Matrix Completion

Suppose someone gave you \$1M for *completing* a table like this...

Matrix Completion

Suppose someone gave you \$1M for *completing* a table like this...

$$
\begin{array}{cc}
 & \textbf{movies} \\
\textbf{users} & \begin{bmatrix}
\star & 0.1 & \star & \star & 0.9 & \star & 0.9 & \star \\
0.5 & 0.4 & \star & \star & \star & 0.6 & \star & 0.6 \\
\star & \star & \star & 0.7 & \star & \star & \star & 0.2 \\
0.4 & 0.2 & 0.9 & \star & \star & \star & \star & \star \\
\star & \star & 0.7 & 0.7 & \star & \star & 0.3 & \star
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\end{array}
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rating user i gives to movie j

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Key insight:

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Key insight: only a few factors may explain users' tastes (genre, actors, ads, ...)

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ratings

$X^\star$

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rating user i gives to movie j

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ratings

$$\underbrace{X^\star}_{\text{ratings}} = \widetilde{U} \, \widetilde{V}$$

Matrix Completion

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influence of factors on users' tastes

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rating user i gives to movie j

<u>Key insight:</u> only a few factors may explain users' tastes (genre, actors, ads, ...)

ratings

characterization of movies

$$X^\star = \widetilde{U} \, \widetilde{V}$$

influence of factors on users' tastes

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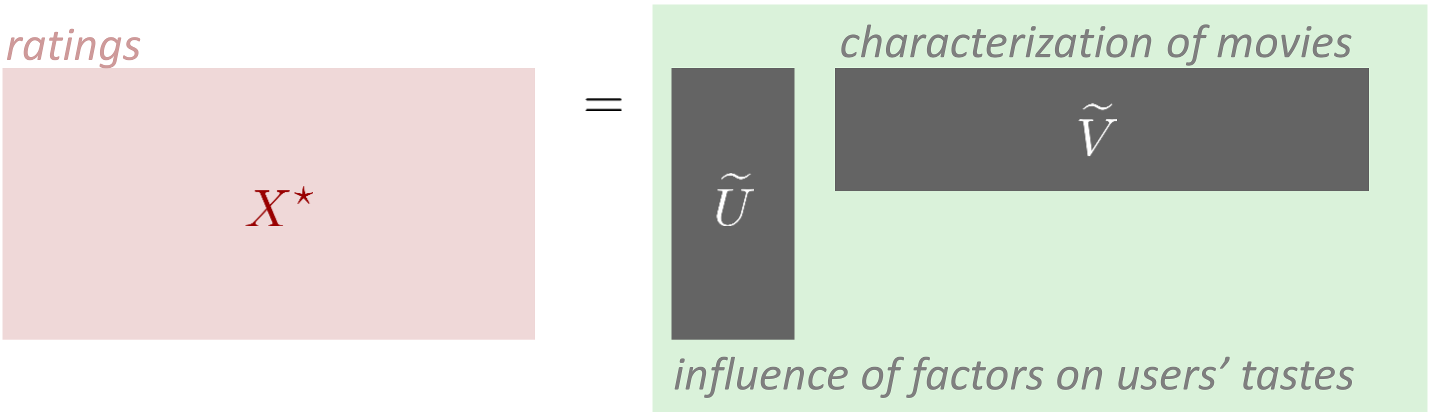

Fewer degrees of freedom

Singular value decomposition:

Singular value decomposition: any real $m \times n$ matrix can be decomposed as

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$$A = U\Sigma V^\top$$

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$$A = U\Sigma V^\top = \underbrace{\begin{bmatrix} | & & | \\ u_1 & \cdots & u_k \\ | & & | \end{bmatrix}}_{m \times k} \underbrace{\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix}}_{k \times k} \underbrace{\begin{bmatrix} - & v_1^\top & - \\ & \vdots & \\ - & v_k^\top & - \end{bmatrix}}_{k \times n}$$

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orthogonal *orthogonal*

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orthogonal *orthogonal*

rank

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orthogonal — rank — orthogonal

$$= \underbrace{\begin{bmatrix} | & & | & | & & | \\ u_1 & \cdots & u_k & u'_{k+1} & \cdots & u'_m \\ | & & | & | & & | \end{bmatrix}}_{m\times m} \underbrace{\left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & 0_{k\times(n-k)} \\ 0 & 0 & \cdots & \sigma_k & \\ \hline & & 0_{(m-k)\times k} & & 0_{(m-k)\times(n-k)} \end{array}\right]}_{m\times n} \underbrace{\begin{bmatrix} - & v_1^\top & - \\ & \vdots & \\ - & v_k^\top & - \\ - & {v'_1}^\top & - \\ & \vdots & \\ - & {v'_k}^\top & - \end{bmatrix}}_{n\times n}$$

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orthogonal

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sparse vector if matrix is low-rank

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orthogonal *rank* *orthogonal*

$$= \underbrace{\begin{bmatrix} | & & | & | & & | \\ u_1 & \cdots & u_k & u'_{k+1} & \cdots & u'_m \\ | & & | & | & & | \end{bmatrix}}_{m\times m} \underbrace{\left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & 0_{k\times(n-k)} \\ 0 & 0 & \cdots & \sigma_k & \\ \hline & 0_{(m-k)\times k} & & & 0_{(m-k)\times(n-k)} \end{array}\right]}_{m\times n} \underbrace{\begin{bmatrix} - & v_1^\top & - \\ & \vdots & \\ - & v_k^\top & - \\ - & {v'_1}^\top & - \\ & \vdots & \\ - & {v'_k}^\top & - \end{bmatrix}}_{n\times n}$$

sparse vector if matrix is low-rank

$$= \underbrace{U\Sigma^{\frac{1}{2}}}_{m\times k}\underbrace{\Sigma^{\frac{1}{2}}V^\top}_{k\times n}$$

$$X^{\star} = \underbrace{U\Sigma^{\frac{1}{2}}}_{m\times k}\underbrace{\Sigma^{\frac{1}{2}}V^{\top}}_{k\times n}$$

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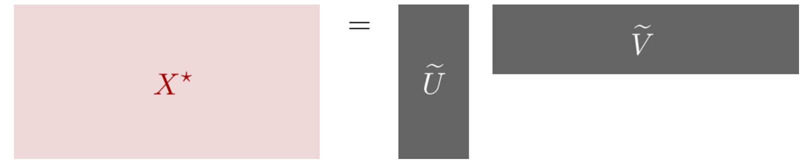

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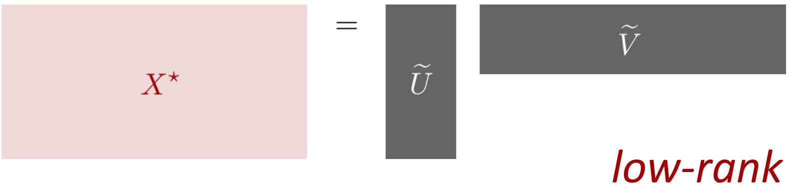

$$X^\star = \underbrace{U\Sigma^{\frac{1}{2}}}_{m\times k}\underbrace{\Sigma^{\frac{1}{2}}V^\top}_{k\times n}$$

$$X^\star = \widetilde{U}\,\widetilde{V}$$

low-rank

Our problem

$$\begin{array}{cc} & \text{movies} \\ \text{users} & \begin{bmatrix} \star & 0.1 & \star & \star & 0.9 & \star & 0.9 & \star \\ 0.5 & 0.4 & \star & \star & \star & 0.6 & \star & 0.6 \\ \star & \star & \star & 0.7 & \star & \star & \star & 0.2 \\ 0.4 & 0.2 & 0.9 & \star & \star & \star & \star & \star \\ \star & \star & 0.7 & 0.7 & \star & \star & 0.3 & \star \end{bmatrix} \end{array}$$

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Our problem

$$\begin{array}{ll} \underset{X\in\mathbb{R}^{m\times n}}{\text{minimize}} & \text{rank}(X) \\ \text{subject to} & X_{ij} = a_{ij}\,, \quad (i,j)\in\mathcal{O} \end{array}$$

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low-rank

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observed entries

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low-rank

Our problem

$$\begin{array}{ll} \underset{X\in\mathbb{R}^{m\times n}}{\text{minimize}} & \text{rank}(X) \\ \text{subject to} & X_{ij} = a_{ij}\,, \quad (i,j)\in\mathcal{O} \end{array}$$

observed entries

nonconvex

$$\begin{array}{cc} & \text{movies} \\ \text{users} & \begin{bmatrix} \star & 0.1 & \star & \star & 0.9 & \star & 0.9 & \star \\ 0.5 & 0.4 & \star & \star & \star & 0.6 & \star & 0.6 \\ \star & \star & \star & 0.7 & \star & \star & \star & 0.2 \\ 0.4 & 0.2 & 0.9 & \star & \star & \star & \star & \star \\ \star & \star & 0.7 & 0.7 & \star & \star & 0.3 & \star \end{bmatrix} \end{array}$$

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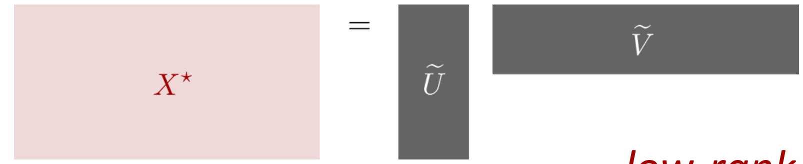

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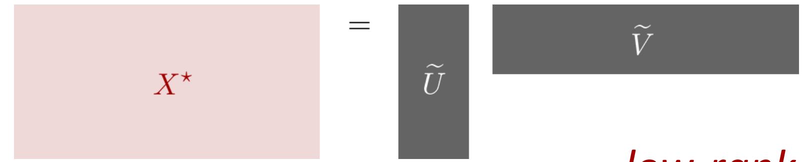

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Theorem [Chandrasekaran et al. 12']

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$$\boxed{p \geq 3k\,(m+n-k)+1} \quad \Longrightarrow \quad \begin{array}{rll} X^\star = & \underset{X}{\mathrm{argmin}} & \|X\|_\star \\ & \text{s.t.} & \mathrm{tr}(XM_l) = a_l\,, \quad l = 1,\ldots,p \end{array}$$

w.h.p.

Experiments

$$\begin{array}{ll} \underset{X \in \mathbb{R}^{m \times n}}{\text{minimize}} & \|X\|_\star \\ \text{subject to} & \operatorname{tr}(X M_l) = a_l\,, \quad l = 1, \ldots, p \end{array}$$

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$X^\star \ : \ 30 \times 30$

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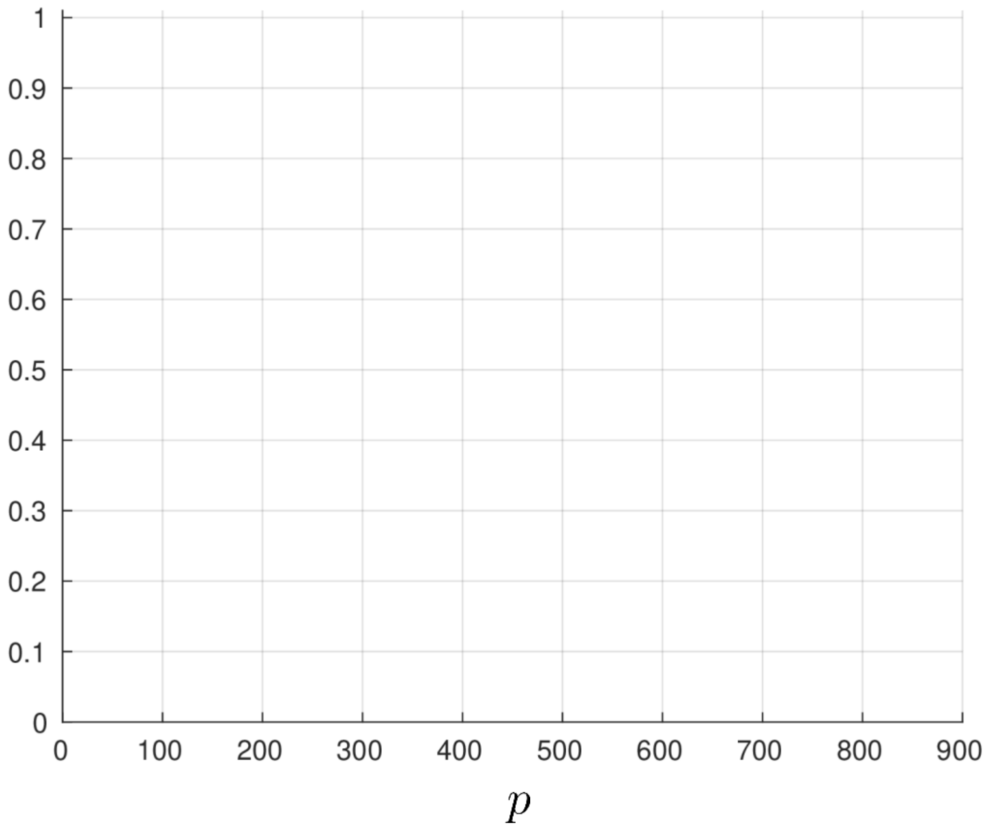
Sucess rate (20 trials)



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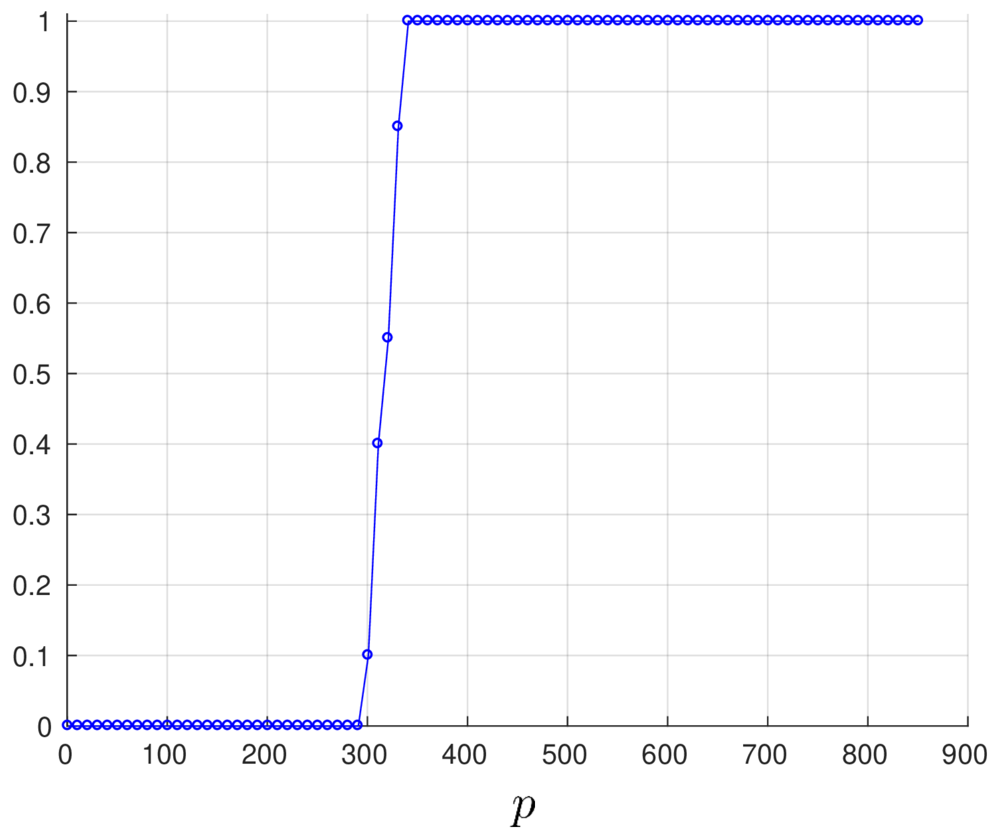

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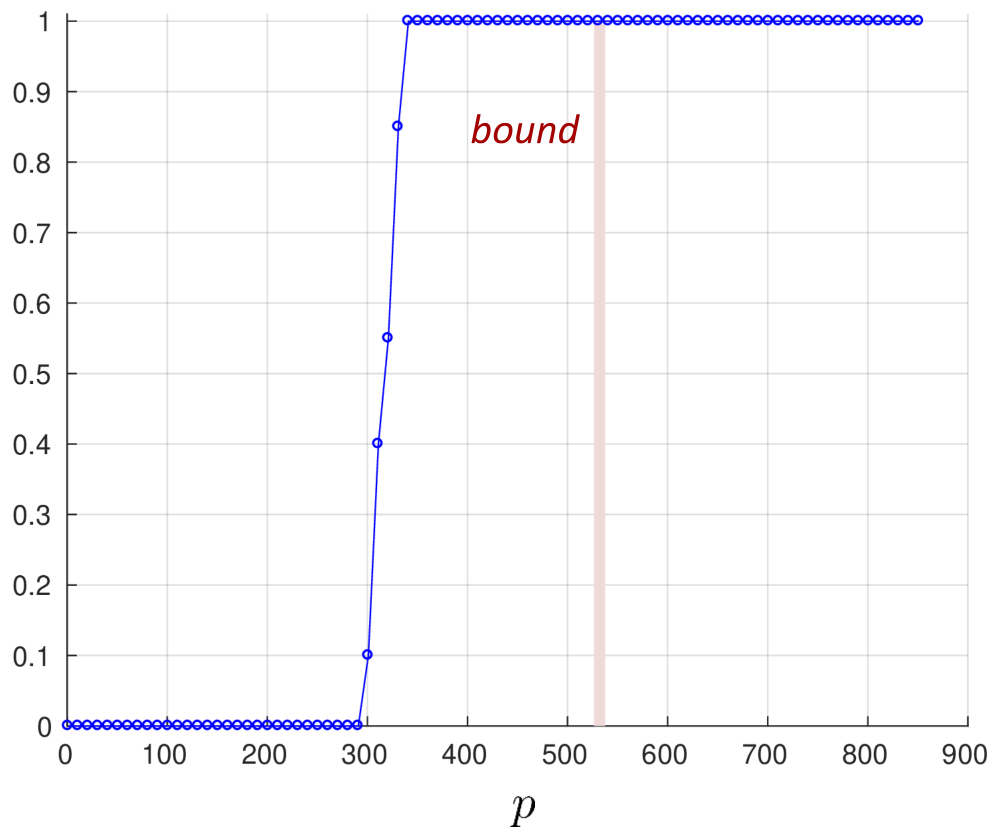

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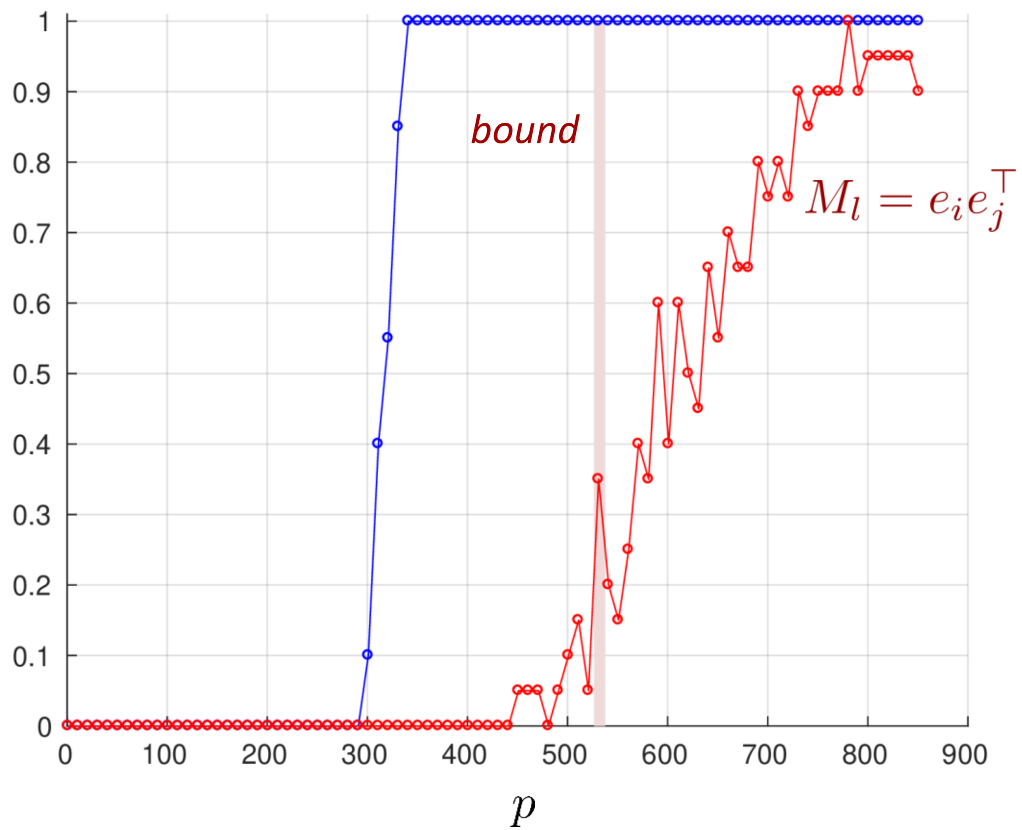

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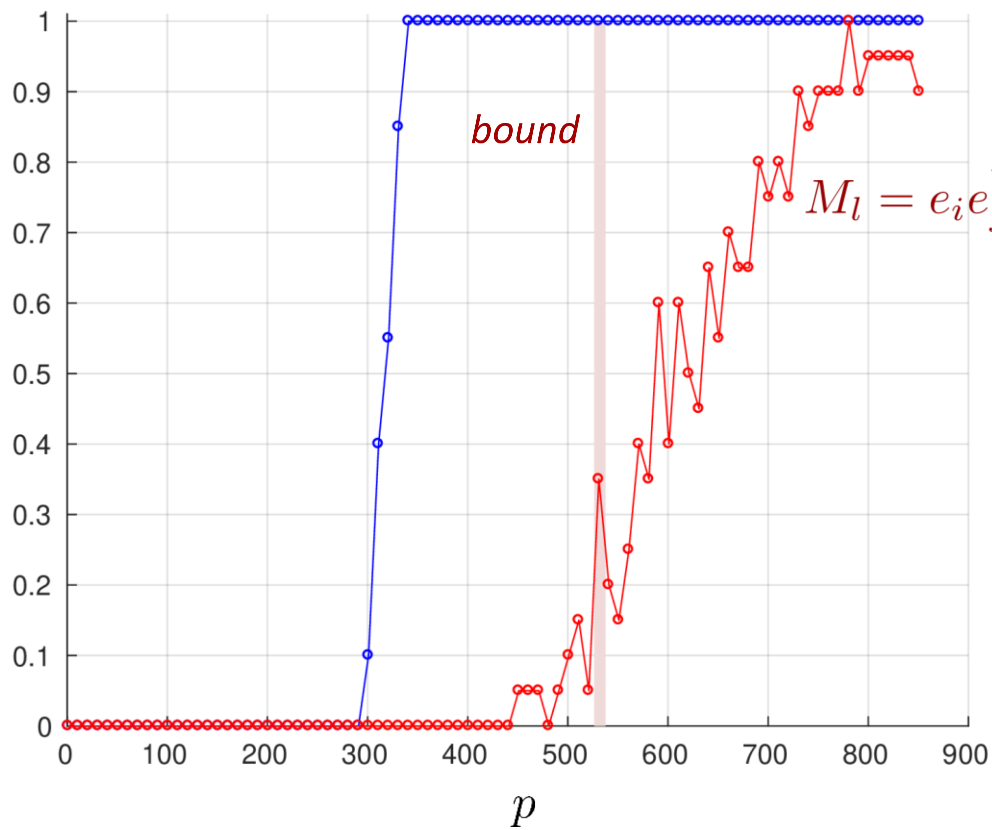

$M_l = e_i e_j^\top$ — all-zeros, and 1 at random entry

Conclusions

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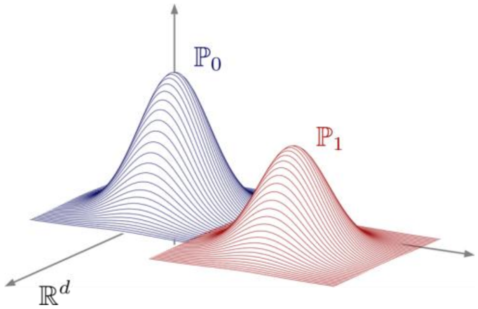

- Structure is key in *high-dimensional* problems

Conclusions

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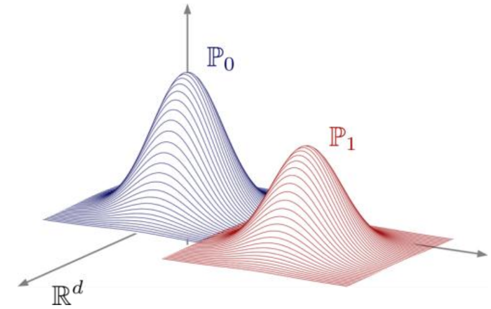

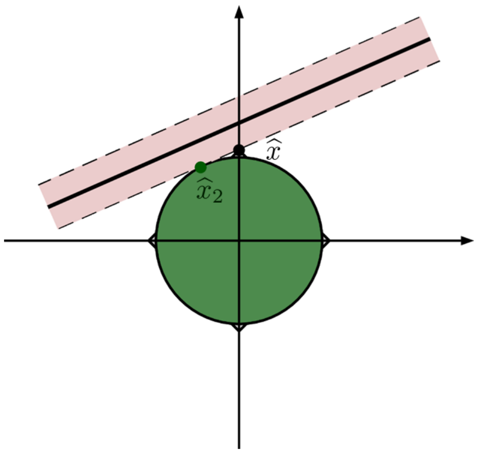

Conclusions

- Structure is key in *high-dimensional* problems
- *Sparsity* encodes several types of structure
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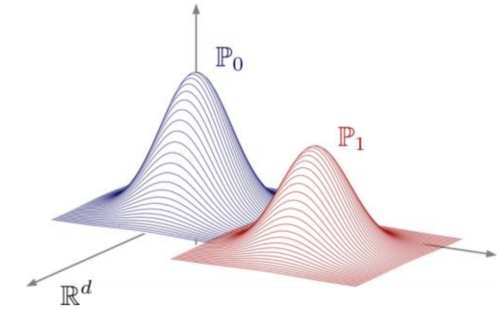

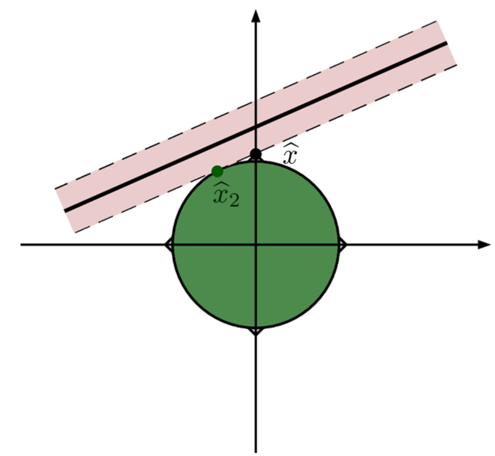

City	Police funding/ resident ($/ya)	% with 4 years high-school (+25)	% not in high-school (16-19)	% in college (18-24)	% with 4+ years college (25+)	Crimes/million
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7	34	68	12	24	14	484
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⋮	⋮	⋮	⋮	⋮	⋮	⋮
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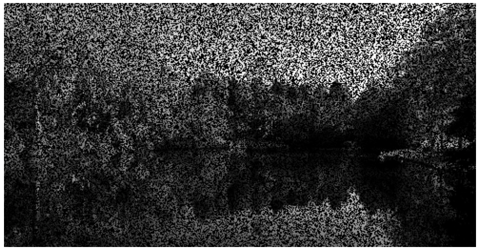

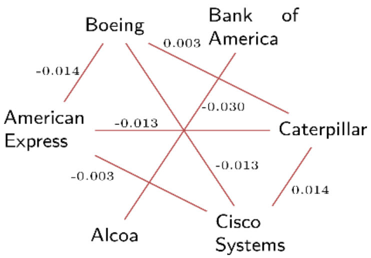

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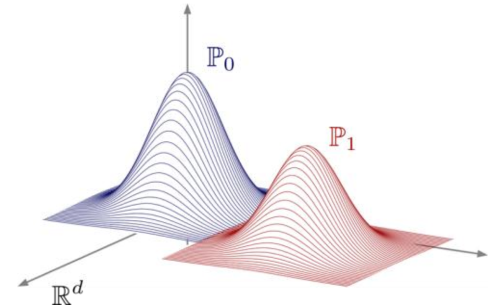

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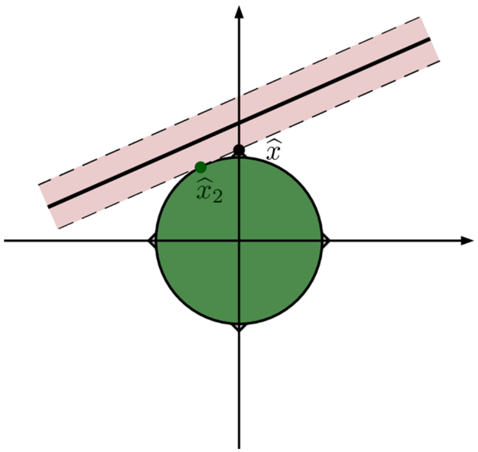

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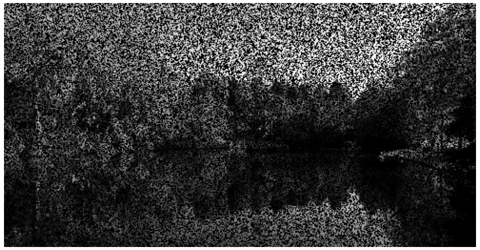

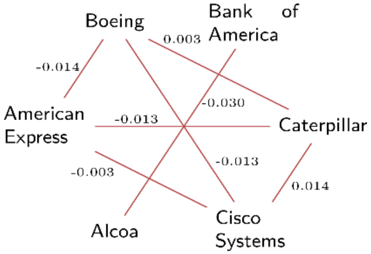

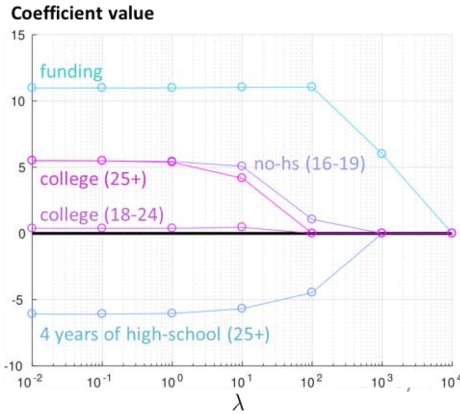

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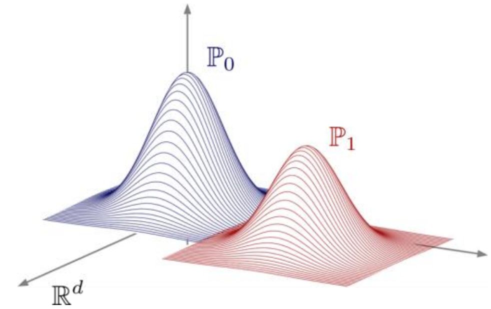

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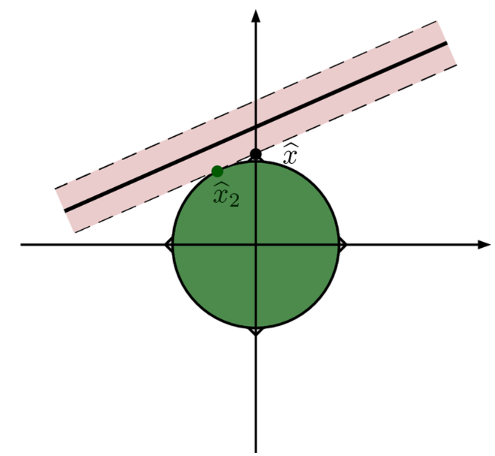

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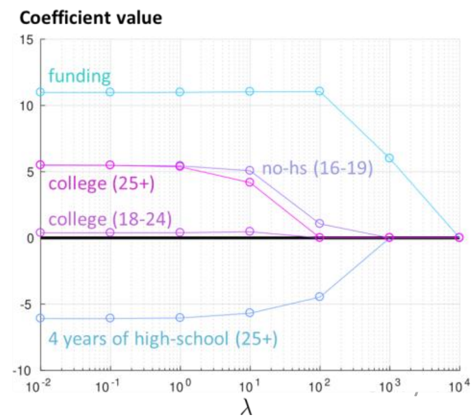

- Didn’t cover: optimization theory and *algorithms*

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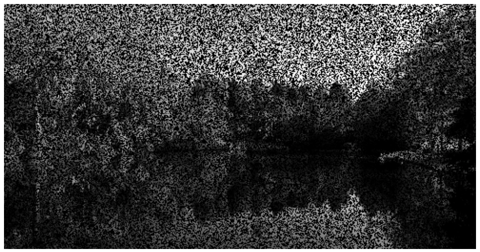

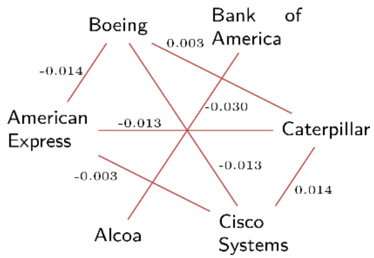

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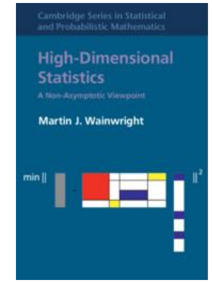

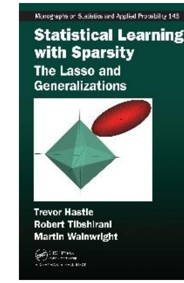

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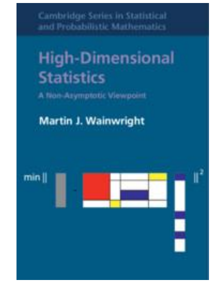

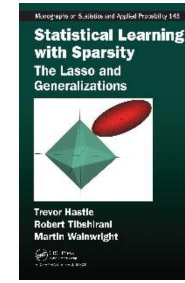

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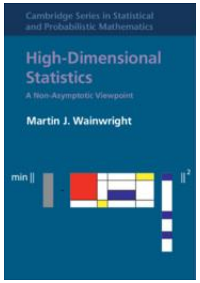

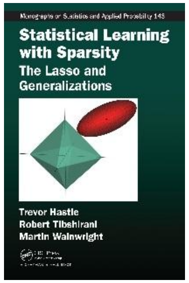

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Code & presentation

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